On Constrained Dual Recoverability Theorems[*]

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Abstract

In a recent paper ["Paretian Welfare Judgements and Bergsonian Social Choice," *Economic Journal*, Vol. 109, 1999, pp. 204-220], Suzumura proposed a possible way of relating the two schools of "new" welfare economics. According to his proposal, the logical possibility of the Paretian "new" welfare economics can be reduced to the *constrained dual choice-functional recoverability* of the Pareto-compatible Bergson-Samuelson social welfare ordering by means of the Pareto-compatible and *consistent* sub-relations thereof. He also identified the necessary and sufficient condition for this crucial property to hold. However, he posed but left open the problem of the constrained dual choice-functional recoverability of the Pareto-compatible Bergson-Samuelson social welfare ordering by means of the Pareto-compatible and *transitive* sub-relations thereof. The first purpose of this paper is to settle this open question. The second purpose of this paper is to pose and settle related problems of the *constrained dual relational recoverability* of the Pareto-compatible Bergson-Samuelson social welfare ordering by means of the Pareto-compatible and either consistent or transitive sub-relations thereof.

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1 Introduction

A binary relation on the universal set of alternatives is said to be *compatible* with another binary relation on the same universal set of alternatives if all the pairwise information conveyed by the former relation are fully consistent with those conveyed by the latter relation. Recent years have witnessed several notable developments in our understanding on the exact relationships which hold between a binary relation and its compatible sub-relations.

To begin with, building on the work of Banerjee and Pattanaik (1996) and Donaldson and Weymark (1998), who started from a quasi-ordering and examined the exact retrievability of all the information thereby conveyed in terms of its compatible ordering extensions, Suzumura and Xu (2002) started from an ordering and examined the exact retrievability of all the information thereby conveyed in terms of the compatible sub-relations thereof. To precisely capture their problems, Suzumura and Xu (2002) proposed two notions, viz., *dual choice-functionally recoverability* and *dual relational recoverability*. An ordering is *dual choice-functionally recoverable* in terms of compatible sub-relations thereof if and only if, for each and every opportunity set, the greatest set in accordance with the given ordering can be exactly retrieved by defining the maximal sets in accordance with each and every compatible sub-relations thereof, and taking their set-theoretical intersection. Likewise, an ordering is *dual relational recoverable* in terms of compatible sub-relations thereof if and only if the ordering exactly coincides with the set-theoretical union of all the compatible sub-relations thereof.

In many contexts, the class of admissible compatible sub-relations is *constrained* by some conditions emanating naturally from the nature of the problem at hand, along with the condition of logical coherence such as transitivity and consistency. In such cases, the problems identified by Suzumura and Xu (2002), may be called the *constrained dual choice-functional recoverability* and *constrained dual relational recoverability*. In this context, there is a recent development in relation to one specific interpretation of the logical nature of "new" welfare economics proposed by Suzumura (1999). Recollect that there are two identifiable schools of thought within the portmanteau catchword of "new" welfare economics. On the one hand, the *compensationist school* a la Kaldor (1939), Hicks (1940), Scitovsky (1941), Samuelson (1950) and Gorman (1955) endeavoured to expand the applicability of the Pareto principle by introducing hypothetical compensation payments between gainers and losers from

a change in economic policy. The gist of this school of thought was best captured by Graaff (1957, pp.84-85), according to whom "[t]he compensation tests all spring from a desire to see what can be said about social welfare ... without making interpersonal comparisons of well-being ... . They have a common origin in Pareto's definition of an increase in social welfare ... but they are extended to situations in which some people are made worse off." On the other hand, the *Bergson-Samuelson school* a la Bergson (1938) and Samuelson (1947, Chapter 8; 1981) introduced the concept of a *social welfare function* with the purpose of going beyond the Pareto quasi-ordering and implementing an ethical norm in the form of a full social welfare ordering. In Arrow's (1951, p.108) parlance, "[the] 'new welfare economics' says nothing about choices among Pareto-optimal alternatives. The purpose of social welfare function was precisely to extend the unanimity quasi-ordering to a full social ordering." As a contrivance for extending the applicability of the Pareto principle to the situations of interpersonal conflict, these two schools have something in common, but their exact relationship seems to have been left unexplored in the literature. Suzumura (1999) proposed one specific way of connecting the two schools of "new" welfare economics by reducing the logical nature of the Paretian "new" welfare economics to the possibility of constrained dual choice-functional recoverability of the Pareto-compatible Bergson-Samuelson social welfare ordering by means of the Pareto-compatible and *consistent* sub-relations thereof. In so doing, he also identified the necessary and sufficient condition for this property to hold. However, he posed but left open the problem of the constrained dual choice-functional recoverability of the Pareto-compatible Bergson-Samuelson social welfare ordering by means of the Pareto-compatible and *transitive* sub-relations thereof.

The purpose of this paper is two-fold. First, we settle the above open question. Secondly, we pose and settle the related problems of the constrained dual relational recoverability of the Pareto-compatible Bergson-Samuelson social welfare ordering by means of the Pareto-compatible and either consistent or transitive sub-relations thereof.

The organization of this paper is as follows. In Section 2, we lay down basic notations and definitions. Section 3 formulates the problems of constrained dual choice-functional and dual relational recoverability of an ordering. Section 4 presents our main results and their proofs. Some concluding remarks are given in Section 5.

2 Basic Notations and Definitions

Let X be the universal set of alternatives with $3 \leq \#X$. A binary relation on X is a subset R of $X \times X$. For all $x, y \in X$, $(x, y) \in R$ may be alternatively written as xRy. An *ordering* R on X is a binary relation that satisfies *reflexivity* : for all $x \in X, (x, x) \in R$, *transitivity* : for all $x, y, z \in X, [(x, y) \in R$ and $(y, z) \in R]$ implies $(x, z) \in R$, and *completeness* : for all distinct $x, y \in X$, $(x, y) \in R$ or $(y, x) \in R$. When R satisfies reflexivity and transitivity, we say that R is a *quasi-ordering*.

For any binary relation R on X, let $P(R)$ and $I(R)$ denote, respectively, the *asymmetric part* of R and the *symmetric part* of R, which are defined by $P(R) = \{(x, y) \in X \times X | (x, y) \in R \ \& \ (y, x) \notin R\}$ and $I(R) = \{(x, y) \in X \times X | (x, y) \in R \ \& \ (y, x) \in R\}$. When R denotes a weak preference relation on X, viz., $(x, y) \in R$ means that x is at least as good as y, $P(R)$ and $I(R)$ denote, respectively, the strict preference relation and the indifference relation corresponding to R.

For any binary relation R and any subset S of X, an element $x \in S$ is an *R-maximal element* of S if $(y, x) \notin P(R)$ holds for all $y \in S$. The set of all R-maximal elements of S is the *R-maximal set* of S, to be denoted by $M(S, R)$. Likewise, an element $x \in S$ is an *R-greatest element* of S if $(x, y) \in R$ holds for all $y \in S$. The set of all R-greatest elements of S is the *R-greatest set* of S, to be denoted by $G(S, R)$.

For any binary relation R on X, a binary relation R^* on X is called an *extension* of R if and only if $R \subset R^*$ and $P(R) \subset P(R^*)$ hold. When R^* is an extension of R, R is called a *compatible sub-relation* of R^*.

A binary relation R is *consistent* if and only if there exists no finite subset $\{x^1, x^2, \cdots, x^t\}$ of X, where $2 \leq t < +\infty$, such that $(x^1, x^2) \in P(R), (x^2, x^3) \in R, \cdots, (x^t, x^1) \in R.$[1] A transitive binary relation is consistent. However, the converse implication does not hold in general.

The following result is due to Suzumura (1976; 1983, Chapter 1), which is a generalization of Szpilrajn's (1930) classical extension theorem.

Lemma 2.1. A binary relation R has an ordering extension if and only if it is consistent.

Let $\Omega(X)$ denote the set of all reflexive and consistent binary relations on X. Also, for any ordering R on X, let $\Theta_0(R)$ denote the set of all compatible sub-relations of R. It follows that $\Theta_0(R) \subset \Omega(X)$ holds for any ordering R on X. By definition,

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[1] The concept of consistency was originally introduced by Suzumura (1976).

$R \in \Theta_0(R)$ holds for any ordering R on X. In view of this fact, a binary relation $Q \in \Theta(R) := \Theta_0(R) \setminus \{R\}$ is called a *compatible strict sub-relation* of R. R is then called a *strict extension* of Q.

3 Constrained Dual Choice-Functional and Dual Relational Recoverability

To prepare the stage of our analysis, let $N = \{1, 2, \cdots, n\}$ be the set of all individuals in the society, where $2 \leq n < +\infty$. Each individual $i \in N$ is assumed to have a weak preference relation R_i on X, the set of all social alternatives, which is an ordering on X. Thus, xR_iy holds if and only if x is judged by i to be at least as good as y. By definition, $P(R_i)$ and $I(R_i)$ stand for i's *strict preference relation* and *indifference relation*, respectively.

Given a list of individual preference orderings $\mathbf{R}^N = (R_1, R_2, \cdots, R_n)$, to be called a *profile* for short, we define the *Pareto quasi-ordering* $\rho(\mathbf{R}^N)$ by

(1) $\quad \rho(\mathbf{R}^N) = \cap_{i \in N} R_i.$

Let R be an ordering on X, to be interpreted as a Bergson-Samuelson social welfare ordering.[2] Throughout this paper, it is assumed that the Bergson-Samuelson social welfare ordering R is Pareto-compatible, so that $\rho(\mathbf{R}^N) \subset R$ and $P(\rho(\mathbf{R}^N)) \subset P(R)$. For any $\mathbf{R}^N$ and any R, let $\Omega(\mathbf{R}^N, R)$ denote the set of all partial welfare judgements, which are strict extensions of $\rho(\mathbf{R}^N)$ as well as strict sub-relations of R:

(2) $\quad \Omega(\mathbf{R}^N, R) := \{Q \subset X \times X | \, \rho(\mathbf{R}^N) \in \Theta(Q) \; \& \; Q \in \Theta(R)\}.$

Let $\mathcal{Q}(X)$ be the set of all quasi-orderings on X and define the set $\Omega^*(\mathbf{R}^N, R)$ by

(3) $\quad \Omega^*(\mathbf{R}^N, R) := \mathcal{Q}(X) \cap \Omega(\mathbf{R}^N, R).$

By construction, $Q \in \Omega^*(\mathbf{R}^N, R)$ holds if and only if (i) Q is a quasi-ordering on X, (ii) Q is a strict extension of the Pareto quasi-ordering $\rho(\mathbf{R}^N)$, and (iii) Q is a strict sub-relation of the Pareto-compatible Bergson-Samuelson social welfare ordering R.

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[2]See Arrow (1983) and Samuelson (1981) for the concept of a Bergson-Samuelson social welfare ordering.

At this juncture of the discussion, it may be worthwhile to point out that filling in the set $\Omega^*(\mathbf{R}^N, R)$ is easier said than done in the analytical framework of ordinal and interpersonally non-comparable information and nothing else. In the first place, there are many interesting proposals for generating Pareto-compatible quasi-orderings, e.g., Suppes (1966), Sen (1970, Chapter 9 & Chapter 9*), Blackorly and Donaldson (1977), and Madden (1996), but they require in common the informational basis which goes beyond ordinal and interpersonally non-comparable information. In the second place, the compensation principles of Kaldor (1939), Hicks (1940), Scitovsky (1941) and Gorman (1955) are based on ordinal and interpersonally non-comparable information, but there are situations where they fail to generate consistent welfare judgements. In the third place, the compensation principle due to Samuelson (1950), which is based on the uniform outward shift of utility possibility frontiers, guarantees transitivity of strict welfare judgements. However, there are situations where not only the Samuelson quasi-ordering fails to be an extension of the Pareto quasi-ordering, but also there exists no Bergson-Samuelson social welfare ordering which subsumes the Pareto quasi-ordering as well as the Samuelson quasi-ordering. See Figure 1 for an example to this effect[3].

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Insert Figure 1 around here
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Let K be the set of all non-empty subsets of X. It is intended that each and every $S \in K$ denotes a *social opportunity set*. We are now ready to introduce formally the concept of constrained dual choice-functional and dual relational recoverability.

Definition 3.1. Let R be a Pareto-compatible social welfare ordering on X. R is *constrained dual choice-functionally recoverable in terms of* $\Omega(\mathbf{R}^N, R)$ if and only if $G(S, R) = \cap_{Q \in \Omega(\mathbf{R}^N, R)} M(S, Q)$ holds for all $S \in K$.

Definition 3.2. Let R be a Pareto-compatible social welfare ordering on X. R is *constrained dual relationally recoverable in terms of* $\Omega(\mathbf{R}^N, R)$ if and only if $R = \cup_{Q \in \Omega(\mathbf{R}^N, R)} Q$ holds.

Definition 3.3. Let R be a Pareto-compatible social welfare ordering on X. R is *constrained dual choice-functionally recoverable in terms of* $\Omega^*(\mathbf{R}^N, R)$ if and only if $G(S, R) = \cap_{Q \in \Omega^*(\mathbf{R}^N, R)} M(S, Q)$ holds for all $S \in K$.

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[3]Figure 1 is reproduced from Suzumura (1999).

Definition 3.4. Let R be a Pareto-compatible social welfare ordering on X. R is *constrained dual relationally recoverable in terms of* $\Omega^*(\mathbf{R}^N, R)$ if and only if $R = \cup_{Q \in \Omega^*(\mathbf{R}^N, R)} Q$ holds.

The meaning of these definitions should be clear from our informal discussion in Introduction. For example, a Pareto-compatible Bergson-Samuelson social welfare ordering R is constrained dual choice-functionally recoverable in terms of $\Omega(\mathbf{R}^N, R)$ if and only if, for each specification of a social opportunity set, the greatest set in accordance with the optimization of R can be retrieved by defining the maximal sets with respect to each and every Pareto-compatible sub-relations of R and taking their set-theoretical intersection.

4 Constrained Dual Recoverability Theorems

Our first result is due to Suzumura (1999).

Theorem 4.1. A Pareto-compatible Bergson-Samuelson social welfare ordering R on X is constrained dual choice-functionally recoverable in terms of $\Omega(\mathbf{R}^N, R)$ if and only if $\Omega(\mathbf{R}^N, R) \neq \emptyset$.

As was shown in Suzumura (1999) in terms of a counter-example, however, an ordering R on X being constrained dual choice-functionally recoverable in terms of $\Omega^*(\mathbf{R}^N, R)$ is not equivalent to $\Omega^*(\mathbf{R}^N, R) \neq \emptyset$. Thus, a necessary and sufficient condition for R to be constrained dual choice-functionally recoverable in terms of the Pareto-compatible, reflexive, and transitive sub-relations thereof must be identified anew. Consider the following assumption.

Assumption 1. For any $x, y \in X$, if $[\forall Q \in \Omega^*(\mathbf{R}^N, R) : Q \cap \{(x, y), (y, x)\} = \emptyset]$, then $(x, y) \in I(R)$.

We are now ready to state the first main result of this paper.

Theorem 4.2. A Pareto-compatible Bergson-Samuelson social welfare ordering R is constrained dual choice-functionally recoverable in terms of $\Omega^*(\mathbf{R}^N, R)$, viz.

(4) $G(S, R) = \cap_{Q \in \Omega^*(\mathbf{R}^N, R)} M(S, Q)$

holds for all $S \in K$, if and only if $\Omega^*(\mathbf{R}^N, R) \neq \emptyset$ and Assumption 1 are satisfied.

Proof. To prove the necessity part, assume that (4) holds for all $S \in K$. Clearly

the set $\Omega^*(\mathbf{R}^N, R)$ must be non-empty. Suppose that $x, y \in X$ are such that $Q \cap \{(x, y), (y, x)\} = \emptyset$ for all $Q \in \Omega^*(\mathbf{R}^N, R)$. Let $S^0 := \{x, y\}$. Since $M(S^0, Q) = \{x, y\}$ for all $Q \in \Omega^*(\mathbf{R}^N, R)$, it follows from (4) that $G(S^0, R) = \cap_{Q \in \Omega^*(\mathbf{R}^N, R)} M(S^0, Q) = \{x, y\}$, which implies that $(x, y) \in I(R)$.

To prove the sufficiency part, we have only to show that

(5) $\cap_{Q \in \Omega^*(\mathbf{R}^N, R)} M(S, Q) \subset G(S, R)$

holds for all $S \in K$, since the reverse set-theoretical inclusion follows immediately from Lemma 1 and Lemma 2 in Suzumura (1999). Assume, therefore, that there exist an $S^* \in K$ and an $x^* \in S^*$ such that $x^* \in \cap_{Q \in \Omega^*(\mathbf{R}^N, R)} M(S^*, Q)$ and that $x \notin G(S^*, R)$. Then there exists a $y^* \in S^*$ such that $(y^*, x^*) \in P(R)$. There are two cases to consider.

Case (i): $\forall Q \in \Omega^*(\mathbf{R}^N, R) : Q \cap \{(x^*, y^*), (y^*, x^*)\} = \emptyset$.
Case (ii): $\exists Q^* \in \Omega^*(\mathbf{R}^N, R) : Q^* \cap \{(x^*, y^*), (y^*, x^*)\} \neq \emptyset$.

In Case (i), it follows from Assumption 1 that $(x^*, y^*) \in I(R)$, which contradicts $(y^*, x^*) \in P(R)$. In Case (ii), there are two sub-cases. If $(x^*, y^*) \in Q^* \cap \{(x^*, y^*), (y^*, x^*)\}$, then $(x^*, y^*) \in Q^* \subset R$ in contradiction with $(y^*, x^*) \in P(R)$. If, on the other hand, $Q^* \cap \{(x^*, y^*), (y^*, x^*)\} = \{(y^*, x^*)\}$, we have $(y^*, x^*) \in P(Q^*)$ in contradiction with $x^* \in M(S^*, Q^*)$. This completes the proof. ∎

We have thus settled a problem raised, but left open, in Suzumura (1999). Let us now turn to the related, but distinct problem of constrained dual relational recoverability. Can the Pareto-compatible Bergson-Samuelson social welfare ordering R be constrained dual relationally recoverable in terms of $\Omega(\mathbf{R}^N, R)$? The following theorem, which is the second main result of this paper, provides our answer to this problem.

Theorem 4.3. A Pareto-compatible Bergson-Samuelson social welfare ordering R on X is constrained dual relationally recoverable in terms of $\Omega(\mathbf{R}^N, R)$ if and only if it is constrained dual choice-functionally recoverable in terms of $\Omega(\mathbf{R}^N, R)$.

Proof. To begin with, note that if R is constrained dual relationally recoverable in terms of $\Omega(\mathbf{R}^N, R)$, then $\Omega(\mathbf{R}^N, R)$ must be non-empty. From Theorem 4.1, therefore, R is constrained dual choice-functionally recoverable in terms of $\Omega(\mathbf{R}^N, R)$. We now show the converse, viz., if R is constrained dual choice-functionally recoverable in terms of $\Omega(\mathbf{R}^N, R)$, then it is constrained dual relationally recoverable in terms of $\Omega(\mathbf{R}^N, R)$.

Suppose, therefore, that R is constrained dual choice-functionally recoverable in terms of $\Omega(\mathbf{R}^N, R)$. From Theorem 4.1, $\Omega(\mathbf{R}^N, R) \neq \emptyset$. Note that, for all $Q \in \Omega(\mathbf{R}^N, R)$, $Q \subset R$. Therefore,

$$(6) \quad \cup_{Q \in \Omega(\mathbf{R}^N, R)} Q \subset R.$$

Thus, we have only to show that $R \subset \cup_{Q \in \Omega(\mathbf{R}^N, R)} Q$. Suppose to the contrary that there exist $x, y \in X$ such that $(x, y) \in R$ and yet $(x, y) \notin \cup_{Q \in \Omega(\mathbf{R}^N, R)} Q$. That is, $(x, y) \in R$ but $(x, y) \notin Q$ for all $Q \in \Omega(\mathbf{R}^N, R)$. Let $S^0 = \{x, y\}$. R being constrained dual choice-functionally recoverable in terms of $\Omega(\mathbf{R}^N, R)$, we have $G(S^0, R) = \cap_{Q \in \Omega(\mathbf{R}^N, R)} M(S^0, Q)$. If $(y, x) \notin R$, then $y \notin G(S^0, R)$. It follows that $y \notin M(S^0, Q)$ for some $Q \in \Omega(\mathbf{R}^N, R)$, so that $(x, y) \in P(Q) \subset Q$, a contradiction. Thus, $(x, y) \in I(R)$ and $(y, x) \notin Q$ for all $Q \in \Omega(\mathbf{R}^N, R)$. Thus, $(x, y) \notin Q$ and $(y, x) \notin Q$ for all $Q \in \Omega(\mathbf{R}^N, R)$. Let $Q^* := \rho(\mathbf{R}^N) \cup \{(x, y), (y, x)\}$. Note that, since the pair $\{x, y\}$ is not ranked by the Pareto quasi-ordering, Q^* is a strict extension of $\rho(\mathbf{R}^N)$. If $Q^* \notin \Omega(\mathbf{R}^N, R)$, then $Q^* = R$, which implies that $\{x, y\}$ is the only Pareto unranked pair, which implies, in turn, that $\Omega(\mathbf{R}^N, R)$ is empty. This is a contradiction with the fact that $\Omega(\mathbf{R}^N, R)$ is non-empty, because R is constrained dual choice-functionally recoverable in terms of $\Omega(\mathbf{R}^N, R)$. Therefore, $Q^* \neq R$. That is, $Q^* \in \Omega(\mathbf{R}^N, R)$. In other words, there exists a $Q \in \Omega(\mathbf{R}^N, R)$ such that $(x, y) \in Q$, another contradiction. Thus, for all $(x, y) \in R$, $(x, y) \in \cup_{Q \in \Omega(\mathbf{R}^N, R)} Q$. This, together with (6), implies that $R = \cup_{Q \in \Omega(\mathbf{R}^N, R)} Q$. Therefore, R is constrained dual relationally recoverable in terms of $\Omega(\mathbf{R}^N, R)$. ∎

According to Theorem 4.3, R being constrained dual choice-functionally recoverable in terms of $\Omega(\mathbf{R}^N, R)$ is equivalent to its being constrained dual relationally recoverable in terms of $\Omega(\mathbf{R}^N, R)$. However, the same conclusion cannot be drawn for R being constrained dual choice-functionally recoverable in terms of $\Omega^*(\mathbf{R}^N, R)$ and R being constrained dual relationally recoverable in terms of $\Omega^*(\mathbf{R}^N, R)$. To see this unambiguously, consider the following counter-example.

Example 4.4. Let $X = \{x, y, z\}$ and let $N = \{1, 2\}$. Consider $R_1 = \{(x, x), (y, y), (z, z), (x, y), (y, z), (x, z)\}$ and $R_2 = \{(x, x), (y, y), (z, z), (x, y), (z, y), (z, x)\}$ and the Bergson-Samuelson social welfare ordering is defined by $R = \{(x, x), (y, y), (z, z), (x, y), (x, z), (y, z), (z, y)\}$, which is Pareto-compatible. Then, $\Omega^*(\mathbf{R}^N, R) = \{Q_1\}$, where $Q_1 = \{(x, x), (y, y), (z, z), (x, y), (x, z)\}$. It can be checked that R is constrained dual choice-functionally recoverable in terms of $\Omega^*(\mathbf{R}^N, R)$. However, it is not constrained dual

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relationally recoverable in terms of $\Omega^*(\mathbf{R}^N, R)$, because $R \neq \cup_{Q \in \Omega^*(\mathbf{R}^N, R)} Q = Q_1$. ∎

Thus, something additional must be satisfied in order for R to be constrained dual relationally recoverable in terms of $\Omega^*(\mathbf{R}^N, R)$. Consider the following assumption on $\Omega^*(\mathbf{R}^N, R)$.

Assumption 2. For all $x, y \in X$, if $(x, y) \in I(R)$, then there exists $Q \in \Omega^*(\mathbf{R}^N, R)$ such that $Q \cup \{(x, y), (y, x)\} \in \Omega^*(\mathbf{R}^N, R)$.

We are now ready to state the third main result of this paper.

Theorem 4.5. A Pareto-compatible Bergson-Samuelson social welfare ordering R on X is constrainted dual relationally recoverable in terms of $\Omega^*(\mathbf{R}^N, R)$ if and only if it is constrained dual choice-functionally recoverable in terms of $\Omega^*(\mathbf{R}^N, R)$ and Assumption 2 is satisfied.

Proof. We first show that if R is constrained dual relationally recoverable in terms of $\Omega^*(\mathbf{R}^N, R)$, then it is constrained dual choice-functionally recoverable and Assumption 2 holds. Suppose that R is constrained dual relationally recoverable in terms of $\Omega^*(\mathbf{R}^N, R)$. Clearly, $\Omega^*(\mathbf{R}^N, R)$ is non-empty. From Theorem 4.2, R being constrained dual choice-functionally recoverable in terms of $\Omega^*(\mathbf{R}^N, R)$ is equivalent to the non-emptiness of $\Omega^*(\mathbf{R}^N, R)$ and Assumption 1 being satisfied. Therefore, we have only to show that Assumption 1 and Assumption 2 hold.

For all $x, y \in X$, since R is constrained dual relationally recoverable in terms of $\Omega^*(\mathbf{R}^N, R)$, if $(x, y) \in R$, then there exists $Q^* \in \Omega^*(\mathbf{R}^N, R)$ such that $(x, y) \in Q^*$. Thus, it can never happen that $Q \cap \{(x, y), (y, x)\} = \emptyset$ for all $Q \in \Omega(\mathbf{R}^N, R)$. Therefore, Assumption 1 holds trivially. Now, let $x, y \in X$ be such that $(x, y) \in I(R)$. Then, by the constrained dual relational recoverability of R in terms of $\Omega^*(\mathbf{R}^N, R)$ and the definition of $\Omega^*(\mathbf{R}^N, R)$, there exists $Q^* \in \Omega(\mathbf{R}^N, R)$ such that $(x, y), (y, x) \in Q^*$. Thus, Assumption 2 holds.

To show the converse, assume that R is constrained dual choice-functionally recoverable and Assumption 2 holds. Since $Q \subset R$ for all $Q \in \Omega^*(\mathbf{R}^N, R)$,

(7) $\cup_{Q \in \Omega^*(\mathbf{R}^N, R)} Q \subset R.$

Thus, we have only to show the converse set-theoretical inclusion. Suppose to the contrary that there exist $x, y \in X$ such that $(x, y) \in R$ but $(x, y) \notin \cup_{Q \in \Omega^*(\mathbf{R}^N, R)} Q$. That is, $(x, y) \in R$ but $(x, y) \notin Q$ for all $Q \in \Omega^*(\mathbf{R}^N, R)$. If $(y, x) \in Q$ for some $Q \in \Omega^*(\mathbf{R}^N, R)$, then $(y, x) \in R$, so that $(x, y) \in I(R)$. Soppose, therefore, $(y, x) \notin Q$

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for all $Q \in \Omega^*(\mathbf{R}^N, R)$. Since R is constrained dual choice-functionally recoverable in terms of $\Omega^*(\mathbf{R}^N, R)$, it must be true that $G(\{x, y\}, R) = \cap_{Q \in \Omega^*(\mathbf{R}^N, R)} M(\{x, y\}, Q) = \{x, y\}$. Thus, $(x, y) \in I(R)$. But then, according to Assumption 2, there exists $Q \in \Omega^*(\mathbf{R}^N, R)$ such that $Q \cup \{(x, y), (y, x)\} \in \Omega^*(\mathbf{R}^N, R)$, which is a contradiction. Therefore, for all $(x, y) \in R$, $(x, y) \in \cup_{Q \in \Omega^*(\mathbf{R}^N, R)} Q$. This, together with (7), implies that R is constrained dual relationally recoverable in terms of $\Omega^*(\mathbf{R}^N, R)$. ∎

5 Concluding Remarks

In this paper, the relationship which holds between an ordering and its compatible sub-relations was explored from the viewpoint of constrained dual recoverability. Following Suzumura (1999), we discussed the issue of constrained dual choice-functional recoverability and constrained dual relational recoverability of the Pareto-compatible Bergson-Samuelson social welfare ordering. Together with Suzumura's earlier work, this paper completed the research agenda of the Paretian "new" welfare economics viewed in terms of the possibility of constrained dual choice-functional recoverability and constrained dual relational recoverability of the Pareto-compatible Bergson-Samuelson social welfare ordering by means of the class of Pareto-compatible and consistent or transitive sub-relations thereof.

In concluding this paper, two final remarks seem to be in order. In the first place, the universe of our discourse was assumed to be an astract set, which has no topological and/or linear structures. In the context of many economic models, however, the universe of discourse is endowed with natural topological and/or linear structures, which allow us to talk sensibly about *continuity* and *convexity*. It may not be without interest to see how much extra mileage we can gain with the help of these natural topological and/or linear structures. As a matter of fact, the key concept of extendability of a binary relation has a natural counterpart in topological spaces, and we have a semi-continuity-preserving extension theorem for strict partial orders by Jaffray (1975), and two semi-continuity-preserving extension theorems for reflexive and consistent binary relations by Bossert, Sprumont and Suzumura (2002). However, the exploration of this promising and potentially important issue must be left for another occasion.

In the second place, the scenario of "new" welfare economics designed by Suzumura (1999), and completed in this paper, is not the only scenario one may conceive and perform. Indeed, there is a related, but distinct scenario, which was explored

by Suzumura and Xu (2001)[4]. According to this alternative scenario, the problem of "new" welfare economics is not to ask the coherence of the Pareto-compatible Bergson-Samuelson social welfare ordering, on the one hand, and the class of partial welfare judgements formed by means of various compensation tests, on the other hand; it is rather to construct the Pareto-compatible Bergson-Samuelson social welfare ordering from within, viz., from the class of Pareto-compatible partial welfare judgements formed by means of various compensation tests. The necessary and sufficient conditions for the constructibility of the Pareto-compatible Bergson-Samuelson social welfare ordering are identified in Suzumura and Xu(2001).

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[4]See Chipman and Moore (1978), and Mishan (1980) for yet another scenarios of "new" welfare economics.

References

Arrow, K. J. (1951). *Social Choice and Individual Values*, New York: Wiley; 2nd ed., 1963.

Arrow, K. J. (1983). "Contributions to Welfare Economics," in Brown, E. Cary and R. M. Solow. eds., *Paul Samuelson and Modern Economic Theory*, New York: McGraw-Hill, pp. 15-30.

Banerjee, A. and Pattanaik, P. K. (1996). "A Note on a Property of Maximal Sets and Choice in the Absence of Universal Comparability," *Economics Letters*, Vol. 51, pp. 191–195.

Bergson, A. (1938). "A Reformulation of Certain Aspects of Welfare Economics," *Quarterly Journal of Economics*, Vol. 52, pp. 310-334.

Blackorby, C. and D. Donaldson (1977). "Utility versus Equity: Some Plausible Quasi-Orderings," *Journal of Public Economics*, Vol. 7, pp. 365-381.

Bossert, W., Sprumont, Y. and Suzumura, K. (2002). "Upper Semicontinuous Extensions of Binary Relations," *Journal of Mathematical Economics*, Vol. 37, pp. 231-246.

Chipman, J. and J. Moore (1978). "The New Welfare Economics 1939-1974," *International Economic Review*, Vol. 19, pp. 547-584.

Donaldson, D. and Weymark, J. A. (1998). "A Quasi-Ordering Is the Intersection of Orderings," *Journal of Economic Theory*, Vol. 78, pp. 382–387.

Gorman, W. M. (1955). "The Intransitivity of Certain Criteria Used in Welfare Economics," *Oxford Economic Papers*, Vol. 7. pp. 25-35.

Graaff, J. de V. (1957). *Theoretical Welfare Economics*, London: Cambridge University Press.

Hicks, J. R. (1940). "The Valuation of the Social Income," *Economica*, Vol. 7, pp. 105-124.

Jaffray, J. -Y. (1975). "Semicontinuous Extensions of a Partial Order," *Journal of Mathematical Economics*, Vol. 2, pp. 395-406.

Kaldor, N. (1939). "Welfare Propositions in Economics and Interpersonal Comparisons of Utility," *Economic Journal*, Vol. 49, pp. 549-552.

Madden, P. (1996). "Suppes-Sen Dominance, Generalized Lorenz Dominace and the Welfare Economics of Competitive Equilibrium: Some Examples," *Journal of Public Economics*, Vol. 61, pp. 247-262.

Mishan, E. J. (1980). "The New Welfare Economics: An Alternative View," *International Economic Review*, Vol. 21, pp. 691-705.

Samuelson, P. A. (1947). *Foundations of Economic Analysis,* Cambridge, Mass.: Harvard University Press. Enlarged ed., 1983.

Samuelson, P. A. (1950). "Evaluation of Real National Income," *Oxford Economic Papers*, Vol. 2, pp. 1-29.

Samuelson, P. A. (1981). "Bergsonian Welfare Economics," in S. Rosefielde, ed., *Economic Welfare and the Economics of Soviet Socialism: Essays in Honor of Abram Bergson*, Cambridge, Mass.: Harvard University Press, pp. 223-266.

Scitovsky, T. (1941). "A Note on Welfare Propositions in Economics," *Review of Economic Studies*, Vol. 9, pp. 77-88.

Sen, A. K. (1970). *Collective Choice and Social Welfare*, San Francisco: Holden-Day.

Suppes, P. (1966) "Some Formal Models of Grading Principles," *Synthese*, Vol. 6, pp. 284-306.

Suzumura, K. (1976). "Remarks on the Theory of Collective Choice," *Economica*, Vol. 43, pp. 381–390.

Suzumura, K. (1983). *Rational Choice, Collective Decisions and Social Welfare*, New York: Cambridge University Press.

Suzumura, K. (1999). "Paretian Welfare Judgements and Bergsonian Social Choice," *Economic Journal*, Vol. 109. pp. 204-220.

Suzumura, K. and Xu, Y. (2001)."Constructing a Bergson-Samuelson Social Welfare Ordering from Partial Welfare Judgements," *mimeo*, Georgia State University.

Suzumura, K. and Xu, Y. (2002). "Recoverability of Choice Functions and Binary Relations: Some Duality Results," *Social Choice and Welfare*, forthcoming.

Szpilrajn, E. (1930). "Sur l'extension de l'ordre partiel," *Fundamenta Mathematicae*, Vol. 16, pp. 386-389.