Multi-Profile Intergenerational Social Choice*

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Abstract. In an infinite-horizon setting, Ferejohn and Page showed that any social welfare function satisfying Arrow's axioms and stationarity must be a dictatorship of the first generation. Packel strengthened this result by proving that no collective choice rule generating complete social preferences can satisfy unlimited domain, weak Pareto and stationarity. We prove that this impossibility survives under a domain restriction and without completeness. We propose a more suitable stationarity axiom and show that a social welfare function on a specific domain satisfies this modified version and some standard social choice axioms if and only if it is a chronological dictatorship. *Journal of Economic Literature* Classification No.: D71.

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1 Introduction

As is well-known, the validity of Arrow's celebrated general impossibility theorem (Arrow, 1951; 1963) hinges squarely on the finiteness of population. Fishburn (1970), Sen (1979) and Suzumura (2000) presented their respective method of proving Arrow's theorem and highlighted the crucial role played by the assumption that the population is finite. Kirman and Sondermann (1972) and Hansson (1976) cast a new light on the structure of an Arrovian social welfare function with an infinite population, revealing the structure of decisive coalitions for such a function as an *ultrafilter.* In their analysis, however, there was no explicit consideration of a sequential relationship among the members of an infinite population. It was a pioneering analysis due to Ferejohn and Page (1978) that introduced time explicitly. Time flows only unidirectionally, and two members t and t' of the society, to be called generation t and generation t', are such that generation t' appears in the society after generation t if and only if t is smaller than t'. As a result of introducing this time structure of infinite population, Ferejohn and Page (1978) also opened a new gate towards combining Arrovian social choice theory and the theory of evaluating infinite intergenerational utility streams, which was initiated by Koopmans (1960) and Diamond (1965). In the traditional Koopmans-Diamond framework, the focus is on resource allocations among different generations with fixed utility functions, one for each generation. Thus, multi-profile considerations do not arise. This paper is an attempt to reexamine the Ferejohn-Page analysis of intergenerational social choice theory in a multi-profile setting.

Starting out with Hansson's (1976) result on the ultrafilter structure of the set of decisive coalitions, Ferejohn and Page (1978) proposed a *stationarity* condition in an infinite-horizon multi-profile social choice model and showed that *if* a social welfare function satisfying Arrow's conditions and stationarity exists, generation one must be a dictator. Stationarity as defined by Ferejohn and Page demands that if a common first-period alternative is eliminated from two infinite streams of per-period alternatives, then the resulting continuation streams must be ranked in the same way as the original streams according to the social ranking obtained *for the original profile.* The reason why generation one is the only candidate for a dictator is the unidirectional nature of the flow of time—and, thus, the unequal treatment of generations in the stationarity property. Dictatorships of later generations fail to satisfy stationarity because we cannot reassess our social evaluation after a later period has passed but an earlier period is still present: we can only move forward but not backward in time.

As Ferejohn and Page (1978) noted themselves, the question whether such a social welfare function exists at all was left open by their analysis; what they showed was that *if* a function with the required properties exists, it must be dictatorial with generation one being the dictator. Packel (1980) answered the question Ferejohn and Page left open by establishing a strong impossibility result: even without independence of irrelevant alternatives and without assuming social preferences to be transitive, no collective choice rule can satisfy unlimited domain, weak Pareto and stationarity. Note that Packel (1980) operates within the same framework as Ferejohn and Page (1978) to establish the impossibility. Thus, his result is not an observation in a different setting but, rather, an answer to the question left open by Ferejohn and Page (1978).

In this paper, we first prove that the negative implications of the Ferejohn-Page stationarity condition are actually more far-reaching: even without reflexivity and completeness, there exists no collective choice rule that satisfies unlimited domain, weak Pareto and stationarity. The same conclusion holds if individual preferences are restricted to those that are history-independent. No restrictions whatsoever are imposed on social preferences—they need not be reflexive, complete or transitive. By dropping reflexivity and completeness, we strengthen Packel's impossibility result substantially. It will become clear once we establish the proof of this impossibility why all collective choice rules (including dictatorships) fail to satisfy the required axioms.

Packel's (1980) approach to resolve the impossibility consisted of restricting the domain of a social welfare function to profiles where generation one's preferences are themselves stationary. This allowed him to obtain possibility results in that setting. In contrast, we think that the natural way to formulate a domain restriction in the intertemporal context is to assume that the preferences of each generation are restricted to depend on the outcome for this generation only. In that case, there do exist social welfare functions that satisfy weak Pareto and stationarity but all of them violate Pareto indifference.

We conclude that the version of stationarity employed by Ferejohn and Page (1978) and by Packel (1980) is too demanding and has some counter-intuitive features. In response, we propose what we suggest is a more suitable multi-profile version of stationarity. Our multi-profile stationarity property requires that, for any two streams of per-period alternatives and for any preference profile, if the first-period alternatives are the same in the two streams, then the social ranking of the two streams according to this profile is *the same* as the social ranking that results if the common first-period alternative is removed *along with the preference ordering of generation one.* Note that there is an essential difference between the Ferejohn and Page (1978) and Packel (1980) version of stationarity and our multi-profile version. Ferejohn and Page (1978) and Packel (1980) continue to apply the original profile (including the preference ordering of generation one) even after a common first-period alternative has been removed. In contrast, our version is, in our opinion, more coherent because it applies to situations in which the preference ordering of the first generation is eliminated as well as the common period-one alternative.

Our main result uses multi-profile stationarity to characterize the lexicographic dictatorship in which the generations are taken into consideration in chronological order. The main conclusion is that, although the infinite-population version of Arrow's social choice problem permits, in principle, non-dictatorial rules, these additional possibilities all but vanish if multi-profile stationarity is imposed.

2 Infinite-Horizon Social Choice

Suppose there is a set of per-period alternatives X containing at least three elements, that is, $|X| \geq 3$ where $|X|$ denotes the cardinality of X. These per-period alternatives could be consumption bundles, for example, but we do not restrict attention to one particular interpretation. Let X^∞ be the set of all infinite streams of per-period alternatives $\mathbf{x} = (x_1, x_2, \ldots)$ where, for each generation $t \in \mathbb{N}$, $x_t \in X$ is the period-t alternative experienced by generation t.

The set of all binary relations on X^∞ is denoted by $\mathcal{B}$, and $\mathcal{C}$ is the set of all complete relations on X^∞. Furthermore, the set of all orderings on X^∞ is denoted by $\mathcal{R}$, where an ordering is a reflexive, complete and transitive relation. A social relation is an element R of $\mathcal{B}$. We assume that each generation $t \in \mathbb{N}$ has an ordering $R_t \in \mathcal{R}$. A (preference) profile is a stream $\mathbf{R} = (R_1, R_2, \ldots)$ of orderings on X^∞. The set of all such profiles is denoted by $\mathcal{R}^\infty$.

Let $t \in \mathbb{N}$. For $\mathbf{x} \in X^\infty$, we define the *period-t continuation of* $\mathbf{x}$ as

$$\mathbf{x}_{\geq t} = (x_t, x_{t+1}, \ldots)$$

and, analogously, for $\mathbf{R} \in \mathcal{R}^\infty$, the *period-$t$ continuation of* $\mathbf{R}$ as

$$\mathbf{R}_{\geq t} = (R_t, R_{t+1}, \ldots).$$

Clearly, $\mathbf{x}_{\geq t}$ is an element of X^∞ for any $t \in \mathbb{N}$ and for any $\mathbf{x} \in X^\infty$. This is the case because we can define $\hat{x}_\tau = x_{\tau+t-1}$ for all $\tau \in \mathbb{N}$, which immediately implies that

$$\hat{\mathbf{x}} = (\hat{x}_1, \hat{x}_2, \ldots) = (x_t, x_{t+1}, \ldots) = \mathbf{x}_{\geq t}$$

is a well-defined element of X^∞. The same reasoning applies to preference profiles: any period-t continuation $\mathbf{R}_{\geq t}$ of a profile $\mathbf{R} \in \mathcal{R}^\infty$ is itself an element of $\mathcal{R}^\infty$.

The above definition of continuations of streams of per-period alternatives is that used by Ferejohn and Page (1978) and Packel (1980). Both Ferejohn and Page and Packel adopt an 'absolute' notion of time: Ferejohn and Page (1978, p.272) interpret $\mathbf{x}_{\geq 2}$ and $\mathbf{y}_{\geq 2}$ as the streams that are obtained if $\mathbf{x}$ and $\mathbf{y}$ are "shifted forward one period" and Packel (1980, p.220) assumes that if $(x_1, x_2, x_3, \ldots)$ is an admissible stream, then so is $(x_2, x_3, \ldots)$, saying that "if a certain overall intergenerational program is possible, then moving the program up one generation is also possible." We follow these authors in adopting this absolute interpretation but note that, due to the observation following the above definition of continuation streams, requiring all streams of per-period alternatives and of preferences to be indexed by all positive integers (thus adopting a 'relative' notion of time) would not change our domains and, thus, our results would be unaffected by such a move. What *is* crucial, however, is the way we combine continuation streams of alternatives and of preferences when defining notions of stationarity. It turns out that the formulation employed by Ferejohn and Page and by Packel is significantly different from ours in this respect, as will become clear in the next two sections.

Two subsets of the *unlimited domain* $\mathcal{R}^\infty$ are of importance in this paper. We define the *forward-looking domain* $\mathcal{R}^\infty_F$ by letting, for all $\mathbf{R} \in \mathcal{R}^\infty$, $\mathbf{R} \in \mathcal{R}^\infty_F$ if and only if, for each $t \in \mathbb{N}$, there exists an ordering Q_t on X^∞ such that, for all $\mathbf{x}, \mathbf{y} \in X^\infty$,

$$\mathbf{x} R_t \mathbf{y} \;\Leftrightarrow\; \mathbf{x}_{\geq t} Q_t \mathbf{y}_{\geq t}.$$

Analogously, the *selfish domain* $\mathcal{R}^\infty_S$ is obtained by letting, for all $\mathbf{R} \in \mathcal{R}^\infty$, $\mathbf{R} \in \mathcal{R}^\infty_S$ if and only if, for each $t \in \mathbb{N}$, there exists an ordering $\succeq_t$ on X such that, for all $\mathbf{x}, \mathbf{y} \in X^\infty$,

$$\mathbf{x} R_t \mathbf{y} \;\Leftrightarrow\; x_t \succeq_t y_t.$$

Clearly, we have $\mathcal{R}^\infty_S \subseteq \mathcal{R}^\infty_F \subseteq \mathcal{R}^\infty$. The relation R_t is an ordering on the set of streams X^∞, whereas $\succeq_t$ is an ordering on the set of per-period alternatives X. On selfish domains, the two can be used interchangeably because, by definition, each generation only cares about its own per-period alternatives. Note that the definition of selfish preferences by itself does not prevent *social* preferences from, for example, using $\succeq_1$ to compare per-period alternatives such as x_t and y_t for periods t that are different from period one. This observation justifies the use of the example following the statement of Theorem 1.

For a relation $R \in \mathcal{B}$, the *asymmetric part* $P(R)$ of R is defined by

$$\mathbf{x} P(R) \mathbf{y} \;\Leftrightarrow\; [\mathbf{x} R \mathbf{y} \text{ and } \neg\, \mathbf{y} R \mathbf{x}]$$

for all $\mathbf{x}, \mathbf{y} \in X^\infty$. The *symmetric part* $I(R)$ of R is defined by

$$\mathbf{x} I(R) \mathbf{y} \;\Leftrightarrow\; [\mathbf{x} R \mathbf{y} \text{ and } \mathbf{y} R \mathbf{x}]$$

for all $\mathbf{x}, \mathbf{y} \in X^\infty$. Furthermore, for all $\mathbf{x}, \mathbf{y} \in X^\infty$ and for all $R \in \mathcal{B}$, $R|_{\{\mathbf{x},\mathbf{y}\}}$ is the restriction of R to the set $\{\mathbf{x}, \mathbf{y}\}$.

In the infinite-horizon context studied in this paper, a *collective choice rule* is a mapping $f\colon \mathcal{D} \to \mathcal{B}$, where $\mathcal{D} \subseteq \mathcal{R}^\infty$ with $\mathcal{D} \neq \emptyset$ is the domain of f. The interpretation is that, for a profile $\mathbf{R} \in \mathcal{D}$, $f(\mathbf{R})$ is the social ranking of streams in X^∞. If $f(\mathcal{D}) \subseteq \mathcal{C}$, f is a *complete collective choice rule.* If $f(\mathcal{D}) \subseteq \mathcal{R}$, f is a *social welfare function.*

Arrow (1951; 1963) imposed the axioms of *unlimited domain, weak Pareto* and *independence of irrelevant alternatives* and showed that, in the case of a finite population, the resulting social welfare functions are *dictatorial:* there exists an individual such that, whenever this individual strictly prefers one alternative over another, this strict preference is reproduced in the social ranking, irrespective of the preferences of other members of society. This result is quite robust with respect to the domain considered. For example, replacing unlimited domain with various alternative domain assumptions (such as the free-triple assumption and others that apply to economic environments) preserves Arrow's impossibility result. In this paper, it turns out that a specific domain restriction (particularly, the selfish domain assumption defined below) allows us to circumvent impossibilities.

The axioms relevant in our context are defined as follows.

Unlimited domain. $\mathcal{D} = \mathcal{R}^\infty$.

Forward-looking domain. $\mathcal{D} = \mathcal{R}^\infty_F$.

Selfish domain. $\mathcal{D} = \mathcal{R}^\infty_S$.

Weak Pareto. For all $\mathbf{x}, \mathbf{y} \in X^\infty$ and for all $\mathbf{R} \in \mathcal{D}$,

$$\mathbf{x} P(R_t) \mathbf{y} \ \forall t \in \mathbb{N} \;\Rightarrow\; \mathbf{x} P(f(\mathbf{R})) \mathbf{y}.$$

Pareto indifference. For all $\mathbf{x}, \mathbf{y} \in X^\infty$ and for all $\mathbf{R} \in \mathcal{D}$,

$$\mathbf{x} I(R_t) \mathbf{y} \ \forall t \in \mathbb{N} \;\Rightarrow\; \mathbf{x} I(f(\mathbf{R})) \mathbf{y}.$$

Independence of irrelevant alternatives. For all $\mathbf{x}, \mathbf{y} \in X^\infty$ and for all $\mathbf{R}, \mathbf{R}' \in \mathcal{D}$,

$$R_t|_{\{\mathbf{x},\mathbf{y}\}} = R'_t|_{\{\mathbf{x},\mathbf{y}\}} \ \forall t \in \mathbb{N} \;\Rightarrow\; f(\mathbf{R})|_{\{\mathbf{x},\mathbf{y}\}} = f(\mathbf{R}')|_{\{\mathbf{x},\mathbf{y}\}}.$$

Let $f\colon \mathcal{D} \to \mathcal{R}$ be a social welfare function and let $\mathbf{x}, \mathbf{y} \in X^\infty$. A set $T \subseteq \mathbb{N}$ (also referred to as a *coalition*) is *decisive for* $\mathbf{x}$ *over* $\mathbf{y}$ *for* f if and only if, for all $\mathbf{R} \in \mathcal{D}$,

$$\mathbf{x} P(R_t) \mathbf{y} \; \forall t \in T \;\Rightarrow\; \mathbf{x} P(f(\mathbf{R})) \mathbf{y}.$$

Furthermore, a set $T \subseteq \mathbb{N}$ is *decisive for* f if and only if T is decisive for $\mathbf{x}$ over $\mathbf{y}$ for f for all $\mathbf{x}, \mathbf{y} \in X^\infty$. Clearly, $\mathbb{N}$ is decisive for any social welfare function f that satisfies weak Pareto. If there is a generation $t \in \mathbb{N}$ such that $\{t\}$ is decisive for f, generation t is a *dictator for* f.

Hansson (1976) has shown that if a social welfare function f satisfies unlimited domain, weak Pareto and independence of irrelevant alternatives, then the set of all decisive coalitions for f must be an *ultrafilter.* An ultrafilter on $\mathbb{N}$ is a collection $\mathcal{U}$ of subsets of $\mathbb{N}$ such that

1. $\emptyset \notin \mathcal{U}$;
2. $\forall T \subseteq \mathbb{N}$, $[T \in \mathcal{U} \text{ or } \mathbb{N} \setminus T \in \mathcal{U}]$;
3. $\forall T, T' \in \mathcal{U}$, $T \cap T' \in \mathcal{U}$.

The conjunction of properties 1 and 2 implies that $\mathbb{N} \in \mathcal{U}$ and, furthermore, the conjunction of properties 1 and 3 implies that the disjunction in property 2 is exclusive—that is, T and $\mathbb{N} \setminus T$ cannot both be in $\mathcal{U}$.

An ultrafilter $\mathcal{U}$ is *principal* if and only if there exists a $t \in \mathbb{N}$ such that, for all $T \subseteq \mathbb{N}$, $T \in \mathcal{U}$ if and only if $t \in T$. Otherwise, $\mathcal{U}$ is a *free* ultrafilter. It can be verified easily that if $\mathbb{N}$ is replaced with a finite set, then the only ultrafilters are principal and, therefore, Hansson's theorem reformulated for finite populations reduces to Arrow's (1951; 1963) theorem—that is, there exists an individual (or a generation) t which is a dictator. In the infinite-population case, a set of decisive coalitions that is a principal ultrafilter corresponds to a dictatorship just as in the finite case. Unlike in the finite case, there also exist free ultrafilters but they cannot be defined explicitly; the proof of their existence relies on non-constructive methods in the sense of using variants of the axiom of choice. These free ultrafilters are non-dictatorial. However, social preferences associated with sets of decisive coalitions that form free ultrafilters fail to be continuous with respect to most standard topologies; see, for instance, Campbell (1990; 1992a,b).

3 Stationarity

None of the above-defined axioms invoke the intertemporal structure imposed by our intergenerational interpretation. In contrast, the following *stationarity* property proposed by Ferejohn and Page (1978) is based on the unidirectional nature of time. The intuition underlying stationarity is that if two streams of per-period alternatives agree in the first period, their relative social ranking is the same as that of their respective period-two continuations. To formulate a property of this nature in a multi-profile setting, the profile under consideration for each of the two comparisons must be specified. In Ferejohn and Page's (1978) and Packel's (1980) contributions, the same profile is employed before and after the common first-period alternative is removed. It seems to us that this leads to a rather demanding requirement because the preferences of the first generation continue to be taken into consideration even though the alternatives relevant for this generation have been eliminated. Ferejohn and Page's (1978) stationarity axiom, the underlying idea of which is due originally to Koopmans (1960) in a related but distinct context, is defined as follows.

Stationarity. For all $\mathbf{x}, \mathbf{y} \in X^\infty$ and for all $\mathbf{R} \in \mathcal{D}$, if $x_1 = y_1$, then

$$\mathbf{x} f(\mathbf{R}) \mathbf{y} \;\Leftrightarrow\; \mathbf{x}_{\geq 2} f(\mathbf{R}) \mathbf{y}_{\geq 2}.$$

Ferejohn and Page's (1978) result establishes that *if* there exists a social welfare function f that satisfies unlimited domain, weak Pareto, independence of irrelevant alternatives and stationarity, then f must be such that generation one is a dictator for f. The existence issue itself remained unresolved by their analysis, as they clearly acknowledge. It was Packel (1980, Theorem 1) who answered this open question in the negative by showing that there does not exist any complete collective choice rule that satisfies unlimited domain, weak Pareto and stationarity. Neither transitivity nor independence of irrelevant alternatives are needed to establish this impossibility result. That even dictatorships do not work under the unlimited domain assumption can be seen by examining the proof of our strengthening of Packel's (1980) impossibility result reported in the following theorem; in fact, our proof is modeled after Packel's own proof, but it uses fewer assumptions to establish the impossibility. We show that, in addition to transitivity, reflexivity and completeness can be dropped and, moreover, the impossibility persists even on the forward-looking domain. Note that, however, the result is not true under the selfish domain, as we establish with an example after proving the theorem.

Theorem 1 *There exists no collective choice rule that satisfies forward-looking domain, weak Pareto and stationarity.*

Proof. Suppose f is a collective choice rule that satisfies the axioms of the theorem statement. Let $x, y \in X$ and let, for each generation t, $\succeq_t$ be an antisymmetric ordering on X such that $yP(\succeq_t)x$ for all odd t and $xP(\succeq_t)y$ for all even t. Define a forward-looking profile $\mathbf{R}$ as follows. For all $\mathbf{x}, \mathbf{y} \in X^\infty$, let

$$\mathbf{x}P(R_1)\mathbf{y} \;\Leftrightarrow\; x_1 P(\succeq_1) y_1 \;\text{ or }\; [x_1 = y_1 \;\text{ and } x_3 P(\succeq_1) y_3].$$

Now let, for all $\mathbf{x}, \mathbf{y} \in X^\infty$, $\mathbf{x}R_1\mathbf{y}$ if and only if $\neg\, \mathbf{y}P(R_1)\mathbf{x}$. For all $t \in \mathbb{N} \setminus \{1\}$ and for all $\mathbf{x}, \mathbf{y} \in X^\infty$, let

$$\mathbf{x}R_t\mathbf{y} \;\Leftrightarrow\; x_t \succeq_t y_t.$$

Clearly, the profile thus defined is in $\mathcal{R}_F^\infty$. Now consider the streams

$$\begin{aligned}
\mathbf{x} &= (x, y, x, y, x, y, \ldots) = (x, \mathbf{y});\\
\mathbf{y} &= (y, x, y, x, y, x, \ldots);\\
\mathbf{z} &= (x, x, y, x, y, x, \ldots) = (x, \mathbf{x}).
\end{aligned}$$

Thus, $\mathbf{x}_{\geq 2} = \mathbf{y}$ and $\mathbf{z}_{\geq 2} = \mathbf{x}$. We have $\mathbf{z}P(R_t)\mathbf{x}$ for all $t \in \mathbb{N}$ and, by weak Pareto, $\mathbf{z}P(f(\mathbf{R}))\mathbf{x}$. Stationarity implies $\mathbf{x}P(f(\mathbf{R}))\mathbf{y}$. But $\mathbf{y}P(R_t)\mathbf{x}$ for all $t \in \mathbb{N}$, and we obtain a contradiction to weak Pareto. ■

Clearly, replacing forward-looking domain with unlimited domain does not affect the validity of the above theorem. Furthermore, there is but a single profile used in the proof and, thus, the conclusion of Theorem 1 is preserved by domain expansion; this is not always the case for results in the spirit of Arrow's (1951; 1963) fundamental theorem.

The impossibility can be resolved by replacing forward-looking domain with selfish domain. To construct an explicit example, consider any selfish profile $\mathbf{R} \in \mathcal{R}_S^\infty$. Recall that, by definition of selfish domain, the profile $\mathbf{R}$ of individual orderings defined on the set X^∞ of *streams of per-period alternatives* is in $\mathcal{R}_S^\infty$ if and only if, for each $t \in \mathbb{N}$, there exists an ordering $\succeq_t$ defined on the set X of *per-period alternatives* such that the relative ranking of two streams $\mathbf{x}$ and $\mathbf{y}$ in X^∞ according to R_t is identical to the relative ranking of the period-t alternatives x_t and y_t according to $\succeq_t$. Thus, the selfish profile $\mathbf{R}$ of orderings defined on X^∞ is completely specified once the profile $(\succeq_1, \succeq_2, \ldots)$ consisting of orderings on X is specified. Suppose $(\succeq_1, \succeq_2, \ldots)$ is the profile of orderings on X associated with the selfish profile $\mathbf{R} \in \mathcal{R}_S^\infty$ of orderings on X^∞. We now define a social welfare function f by letting, for all $\mathbf{x}, \mathbf{y} \in X^\infty$ and for all $\mathbf{R} \in \mathcal{R}_S^\infty$, $\mathbf{x} f(\mathbf{R}) \mathbf{y}$ if and only if

$$[x_\tau I(\succeq_1) y_\tau \;\forall\, \tau \in \mathbb{N}] \;\text{ or }\; [\exists t \in \mathbb{N} \;\text{ such that }\; [x_\tau I(\succeq_1) y_\tau \;\forall \tau < t \;\text{ and }\; x_t P(\succeq_1) y_t]]$$

The social welfare function f satisfies selfish domain, weak Pareto and stationarity. However, it does not satisfy Pareto indifference. Intuitively, this is the case because, at the *social* level, generation one's per-period preferences are consulted not only in period one but also in *later* periods, whereas the per-period preferences of other generations do not influence the social comparisons at all. More generally, replacing forward-looking domain with selfish domain and adding Pareto indifference in Theorem 1 produces another impossibility.

Theorem 2 *There exists no collective choice rule that satisfies selfish domain, weak Pareto, Pareto indifference and stationarity.*

Proof. Suppose f is a collective choice rule that satisfies the axioms of the theorem statement. Let $x, y, z \in X$ and let, for each generation $t \in \mathbb{N}$, $\succeq_t$ be an ordering on X such that $zP(\succeq_t)xI(\succeq_t)y$ for all odd t and $zI(\succeq_t)xP(\succeq_t)y$ for all even t. Define a profile $\mathbf{R}$ as follows. For all $\mathbf{x}, \mathbf{y} \in X^\infty$ and for all $t \in \mathbb{N}$, let

$$\mathbf{x} R_t \mathbf{y} \;\Leftrightarrow\; x_t \succeq_t y_t.$$

Clearly, the profile thus defined is in $\mathcal{R}_S^\infty$. Now consider the streams

$$\begin{aligned}
\mathbf{x} &= (z, x, z, x, z, x, \ldots);\\
\mathbf{y} &= (x, y, x, y, x, y, \ldots);\\
\mathbf{z} &= (z, z, x, z, x, z, x, \ldots) = (z, \mathbf{x});\\
\mathbf{w} &= (z, x, y, x, y, x, y, \ldots) = (z, \mathbf{y}).
\end{aligned}$$

Thus, $\mathbf{z}_{\geq 2} = \mathbf{x}$ and $\mathbf{w}_{\geq 2} = \mathbf{y}$. We have $\mathbf{z} I(R_t) \mathbf{w}$ for all $t \in \mathbb{N}$ and, by Pareto indifference, $\mathbf{z} I(f(\mathbf{R})) \mathbf{w}$. Stationarity implies $\mathbf{x} I(f(\mathbf{R})) \mathbf{y}$. But $\mathbf{x} P(R_t) \mathbf{y}$ for all $t \in \mathbb{N}$, and we obtain a contradiction to weak Pareto. ■

Packel's (1980) response to his impossibility result consisted of restricting the domain to profiles that only contain generation-one preferences that are themselves stationary, thus ruling out the type of profile that he used in his impossibility theorem (and that we use in our Theorem 1). In contrast, we think that the selfish domain represents a plausible restriction of preferences in an intergenerational setting, and we therefore propose to amend Ferejohn and Page's (1978) stationarity condition in order to allow for possibility results. One shortcoming we see with stationarity as defined in this section is that it applies to period-two continuations of streams in which common period-one alternatives are eliminated, whereas the original profile is retained *even though the alternatives relevant for generation one are not present in the continuations.* The alternative variant that we introduce in the following section does not suffer from this problem.

4 Multi-Profile Stationarity

In Ferejohn and Page's (1978) stationarity axiom, the same profile $\mathbf{R}$ is applied both before and after the common period-one alternative is eliminated. This seems to us to be rather counter-intuitive and, consequently, we propose the following version that takes this point into consideration by eliminating the (common) first-period component not only from the streams but also from the profile. When combined with selfish domain, this appears to be a natural version of the axiom.

Multi-profile stationarity. For all $\mathbf{x}, \mathbf{y} \in X^\infty$ and for all $\mathbf{R} \in \mathcal{D}$, if $x_1 = y_1$, then

$$\mathbf{x} f(\mathbf{R}) \mathbf{y} \;\Leftrightarrow\; \mathbf{x}_{\geq 2} f(\mathbf{R}_{\geq 2}) \mathbf{y}_{\geq 2}.$$

Note that the continuation profile $\mathbf{R}_{\geq 2}$ is only used in comparing the continuation streams $\mathbf{x}_{\geq 2}$ and $\mathbf{y}_{\geq 2}$ in the definition of multi-profile stationarity. Thus, there is no conflict with the selfish domain assumption.

We now examine the implications of our multi-profile stationarity axiom. In particular, it allows us to characterize the *chronological dictatorship.* This variant of a lexicographic dictatorship consults generation one first but, in the case of its indifference, moves on to consult *generation two* regarding the ranking of two streams, and so on. Thus, there still is a strong dictatorship component but it is not as extreme as that generated by stationarity—and it is compatible with Pareto indifference. Moreover, the chronological dictatorship is a social welfare function and not merely a collective choice rule.

The chronological dictatorship f^{CD} is defined as follows (again, recall that the orderings $\succeq_t$ on X are sufficient to identify the corresponding selfish orderings R_t on X^∞). For all $\mathbf{x}, \mathbf{y} \in X^\infty$ and for all $\mathbf{R} \in \mathcal{R}_S^\infty$, $\mathbf{x} f^{CD}(\mathbf{R}) \mathbf{y}$ if and only if

$$[x_\tau I(\succeq_\tau) y_\tau \;\forall\, \tau \in \mathbb{N}] \;\text{ or }\; [\exists t \in \mathbb{N} \;\text{ such that }\; [x_\tau I(\succeq_\tau) y_\tau \;\forall \tau < t \;\text{ and }\; x_t P(\succeq_t) y_t]].$$

In order to prove a version of Hansson's (1976) theorem that applies to the selfish domain, we require Pareto indifference as an additional axiom. A modification of this nature is required because the selfish domain is not sufficiently rich to generate arbitrary rankings of all streams of alternatives. For instance, whenever we have two streams of per-period alternatives $\mathbf{x}$ and $\mathbf{y}$ such that $x_t = y_t$ for some selfish generation $t \in \mathbb{N}$, this selfish generation *must* declare $\mathbf{x}$ and $\mathbf{y}$ indifferent: a per-period alternative cannot be strictly preferred to itself; in fact, indifference is forced by the conjunction of selfish domain and reflexivity. More precisely, this addition of Pareto indifference to the list of axioms is necessitated by the observation that a fundamental preliminary result—an adaptation of Sen's (1995, p.4) *field expansion lemma* to our selfish domain setting—fails to be true if merely selfish domain, weak Pareto and independence of irrelevant alternatives are imposed.

Our version of the field expansion lemma is stated below. Because we invoke Pareto indifference in addition to the remaining axioms of the statement of the lemma, its proof varies from that of the standard formulation.

Lemma 1 *Let f be a social welfare function that satisfies selfish domain, weak Pareto, Pareto indifference and independence of irrelevant alternatives, and let $T \subseteq \mathbb{N}$. If there exist $\mathbf{x}, \mathbf{y} \in X^\infty$ such that $x_t \neq y_t$ for all $t \in \mathbb{N}$ and T is decisive for $\mathbf{x}$ over $\mathbf{y}$ for f, then T is decisive for f.*

Proof. Let f be a social welfare function that satisfies the axioms of the lemma statement, and let $T \subseteq \mathbb{N}$. Suppose that $\mathbf{x}, \mathbf{y} \in X^\infty$ are such that $x_t \neq y_t$ for all $t \in \mathbb{N}$ and that T is decisive for $\mathbf{x}$ over $\mathbf{y}$ for f. In order to cover all possible cases, we have to establish that T is decisive:

(i) for $\mathbf{x}$ over $\mathbf{z}$ for f for all $\mathbf{z} \in X^\infty \setminus \{\mathbf{x}, \mathbf{y}\}$;

(ii) for $\mathbf{z}$ over $\mathbf{y}$ for f for all $\mathbf{z} \in X^\infty \setminus \{\mathbf{x}, \mathbf{y}\}$;

(iii) for $\mathbf{z}$ over $\mathbf{x}$ for f for all $\mathbf{z} \in X^\infty \setminus \{\mathbf{x}, \mathbf{y}\}$;

(iv) for $\mathbf{y}$ over $\mathbf{z}$ for f for all $\mathbf{z} \in X^\infty \setminus \{\mathbf{x}, \mathbf{y}\}$;

(v) for $\mathbf{z}$ over $\mathbf{w}$ for all distinct $\mathbf{z}, \mathbf{w} \in X^\infty \setminus \{\mathbf{x}, \mathbf{y}\}$;

(vi) for $\mathbf{y}$ over $\mathbf{x}$ for f.

First, note that if there exists $t \in T$ such that $z_t = x_t$ (or $z_t = y_t$ or $z_t = w_t$, depending on which case applies), T is trivially decisive for $\mathbf{x}$ over $\mathbf{z}$ (or for $\mathbf{z}$ over $\mathbf{y}$ or for $\mathbf{z}$ over $\mathbf{x}$ or for $\mathbf{y}$ over $\mathbf{z}$ or for $\mathbf{z}$ over $\mathbf{w}$, respectively) because, by reflexivity and the assumption that preferences are selfish, we must have that $\mathbf{x}$ and $\mathbf{z}$ (or $\mathbf{z}$ and $\mathbf{y}$ or $\mathbf{z}$ and $\mathbf{w}$) are indifferent for any generation $t \in T$ such that $z_t = x_t$ (or $z_t = y_t$ or $z_t = w_t$) and, thus, the implication defining decisiveness is vacuously satisfied for any selfish profile. Thus, we can suppose that, for all $t \in \mathbb{N}$, $z_t \neq x_t$ in (i) and in (iii), $z_t \neq y_t$ in (ii) and in (iv), and $z_t \neq w_t$ in (v). Furthermore, observe that and $x_t \neq y_t$ is assumed throughout (and, in particular, in (vi)) due to the hypothesis of the lemma.

(i) Define an alternative $\mathbf{z}' \in X^\infty$ by letting

$$z'_t = \begin{cases} z_t & \text{if } z_t \neq y_t; \\ z'_t \in X \setminus \{x_t, y_t\} & \text{if } z_t = y_t \end{cases}$$

for all $t \in \mathbb{N}$. By selfish domain, we can define three profiles $\mathbf{R}, \mathbf{R}', \mathbf{R}'' \in \mathcal{R}_S^\infty$ such that

$$\begin{aligned} [\mathbf{x}P(R_t)\mathbf{z} \;\text{ and }\; \mathbf{x}P(R'_t)\mathbf{y}] &\qquad \forall t \in T; \\ [\mathbf{y}P(R'_t)\mathbf{z}' \;\text{ and }\; \mathbf{z}'I(R''_t)\mathbf{z}] &\qquad \forall t \in \mathbb{N}; \\ R''_t|_{\{\mathbf{x},\mathbf{z}'\}} = R'_t|_{\{\mathbf{x},\mathbf{z}'\}} &\qquad \forall t \in \mathbb{N}; \\ R''_t|_{\{\mathbf{x},\mathbf{z}\}} = R_t|_{\{\mathbf{x},\mathbf{z}\}} &\qquad \forall t \in \mathbb{N}. \end{aligned}$$

Because T is decisive for $\mathbf{x}$ over $\mathbf{y}$ for f by assumption, we have $\mathbf{x}P(f(\mathbf{R}'))\mathbf{y}$. By weak Pareto, $\mathbf{y}P(f(\mathbf{R}'))\mathbf{z}'$. By transitivity, $\mathbf{x}P(f(\mathbf{R}'))\mathbf{z}'$. By independence of irrelevant alternatives, $\mathbf{x}P(f(\mathbf{R}''))\mathbf{z}'$. Pareto indifference implies $\mathbf{z}'I(f(\mathbf{R}''))\mathbf{z}$ and, by transitivity, $\mathbf{x}P(f(\mathbf{R}''))\mathbf{z}$. By independence of irrelevant alternatives, it follows that $\mathbf{x}P(f(\mathbf{R}))\mathbf{z}$ and that T is decisive for $\mathbf{x}$ over $\mathbf{z}$ for f.

(ii) Define an alternative $\mathbf{z}' \in X^\infty$ by letting

$$z'_t = \begin{cases} z_t & \text{if } z_t \neq x_t; \\ z'_t \in X \setminus \{x_t, y_t\} & \text{if } z_t = x_t \end{cases}$$

for all $t \in \mathbb{N}$. By selfish domain, we can define three profiles $\mathbf{R}, \mathbf{R}', \mathbf{R}'' \in \mathcal{R}_S^\infty$ such that

$$\begin{aligned} [\mathbf{z}P(R_t)\mathbf{y} \;\text{ and }\; \mathbf{x}P(R'_t)\mathbf{y}] &\qquad \forall t \in T; \\ [\mathbf{z}'P(R'_t)\mathbf{x} \;\text{ and }\; \mathbf{z}'I(R''_t)\mathbf{z}] &\qquad \forall t \in \mathbb{N}; \\ R''_t|_{\{\mathbf{y},\mathbf{z}'\}} = R'_t|_{\{\mathbf{y},\mathbf{z}'\}} &\qquad \forall t \in \mathbb{N}; \\ R''_t|_{\{\mathbf{y},\mathbf{z}\}} = R_t|_{\{\mathbf{y},\mathbf{z}\}} &\qquad \forall t \in \mathbb{N}. \end{aligned}$$

By weak Pareto, the decisiveness of T for $\mathbf{x}$ over $\mathbf{y}$ for f, transitivity and independence of irrelevant alternatives, we obtain $\mathbf{z}'P(f(\mathbf{R}''))\mathbf{y}$. Applying Pareto indifference, transitivity and independence of irrelevant alternatives, it follows that $\mathbf{z}P(f(\mathbf{R}))\mathbf{y}$ and that T is decisive for $\mathbf{z}$ over $\mathbf{y}$ for f.

(iii) Define an alternative $\mathbf{z}' \in X^\infty$ by letting

$$z'_t = \begin{cases} z_t & \text{if } z_t \neq y_t; \\ z'_t \in X \setminus \{x_t, y_t\} & \text{if } z_t = y_t \end{cases}$$

for all $t \in \mathbb{N}$. By selfish domain, we can define three profiles $\mathbf{R}, \mathbf{R}', \mathbf{R}'' \in \mathcal{R}_S^\infty$ such that

$$\begin{aligned} [\mathbf{z}P(R_t)\mathbf{x} \text{ and } \mathbf{z}'P(R'_t)\mathbf{y}] & \quad \forall t \in T; \\ [\mathbf{y}P(R'_t)\mathbf{x} \text{ and } \mathbf{z}'I(R''_t)\mathbf{z}] & \quad \forall t \in \mathbb{N}; \\ R''_t|_{\{\mathbf{x},\mathbf{z}'\}} = R'_t|_{\{\mathbf{x},\mathbf{z}'\}} & \quad \forall t \in \mathbb{N}; \\ R''_t|_{\{\mathbf{x},\mathbf{z}\}} = R_t|_{\{\mathbf{x},\mathbf{z}\}} & \quad \forall t \in \mathbb{N}. \end{aligned}$$

By the decisiveness of T for $\mathbf{z}'$ over $\mathbf{y}$ for f (see (ii)), weak Pareto, transitivity and independence of irrelevant alternatives, we obtain $\mathbf{z}'P(f(\mathbf{R}''))\mathbf{x}$. Pareto indifference, transitivity and independence of irrelevant alternatives together imply that $\mathbf{z}P(f(\mathbf{R}))\mathbf{x}$ and that T is decisive for $\mathbf{z}$ over $\mathbf{x}$ for f.

(iv) Define an alternative $\mathbf{z}' \in X^\infty$ by letting

$$z'_t = \begin{cases} z_t & \text{if } z_t \neq x_t; \\ z'_t \in X \setminus \{x_t, y_t\} & \text{if } z_t = x_t \end{cases}$$

for all $t \in \mathbb{N}$. By selfish domain, we can define three profiles $\mathbf{R}, \mathbf{R}', \mathbf{R}'' \in \mathcal{R}_S^\infty$ such that

$$\begin{aligned} [\mathbf{y}P(R_t)\mathbf{z} \text{ and } \mathbf{x}P(R'_t)\mathbf{z}'] & \quad \forall t \in T; \\ [\mathbf{y}P(R'_t)\mathbf{x} \text{ and } \mathbf{z}'I(R''_t)\mathbf{z}] & \quad \forall t \in \mathbb{N}; \\ R''_t|_{\{\mathbf{y},\mathbf{z}'\}} = R'_t|_{\{\mathbf{y},\mathbf{z}'\}} & \quad \forall t \in \mathbb{N}; \\ R''_t|_{\{\mathbf{y},\mathbf{z}\}} = R_t|_{\{\mathbf{y},\mathbf{z}\}} & \quad \forall t \in \mathbb{N}. \end{aligned}$$

By weak Pareto, the decisiveness of T for $\mathbf{x}$ over $\mathbf{z}'$ for f (see (i)), transitivity and independence of irrelevant alternatives, we obtain $\mathbf{y}P(f(\mathbf{R}''))\mathbf{z}'$. Pareto indifference, transitivity and independence of irrelevant alternatives together imply that $\mathbf{y}P(f(\mathbf{R}))\mathbf{z}$ and that T is decisive for $\mathbf{y}$ over $\mathbf{z}$ for f.

(v) Let two alternatives $\mathbf{z}', \mathbf{w}' \in X^\infty$ be such that

$$z'_t = \begin{cases} z_t & \text{if } z_t \neq x_t; \\ z'_t \in X \setminus \{x_t, w_t\} & \text{if } z_t = x_t \end{cases}$$

for all $t \in T$ and

$$w'_t = \begin{cases} w_t & \text{if } w_t \neq x_t; \\ w'_t \in X \setminus \{x_t, z_t\} & \text{if } w_t = x_t \end{cases}$$

for all $t \in T$. Note that, because $z_t \neq w_t$ for all $t \in T$ in this case, x_t, z'_t and w'_t are pairwise distinct for all $t \in T$. By selfish domain, we can define three profiles $\mathbf{R}, \mathbf{R}', \mathbf{R}'' \in \mathcal{R}_S^\infty$ such that

$$\begin{aligned} [\mathbf{z}P(R_t)\mathbf{w} \text{ and } \mathbf{z}'P(R'_t)\mathbf{x}P(R'_t)\mathbf{w}'] \quad & \forall t \in T; \\ [\mathbf{z}'I(R''_t)\mathbf{z} \text{ and } \mathbf{w}'I(R''_t)\mathbf{w}] \quad & \forall t \in \mathbb{N}; \\ R''_t|_{\{\mathbf{z}',\mathbf{w}'\}} = R'_t|_{\{\mathbf{z}',\mathbf{w}'\}} \quad & \forall t \in \mathbb{N}; \\ R''_t|_{\{\mathbf{z},\mathbf{w}\}} = R_t|_{\{\mathbf{z},\mathbf{w}\}} \quad & \forall t \in \mathbb{N}. \end{aligned}$$

By the decisiveness of T for $\mathbf{z}'$ over $\mathbf{x}$ (see (iii)) and for $\mathbf{x}$ over $\mathbf{w}'$ (see (i)) for f, transitivity and independence of irrelevant alternatives, we obtain $\mathbf{z}'P(f(\mathbf{R}''))\mathbf{w}'$. Now Pareto indifference, transitivity and independence of irrelevant alternatives together imply that $\mathbf{z}P(f(\mathbf{R}))\mathbf{w}$ and that T is decisive for $\mathbf{z}$ over $\mathbf{w}$ for f.

(vi) Let $\mathbf{z} \in X^\infty$ be such that $z_t \notin \{x_t, y_t\}$ for all $t \in T$. By selfish domain, we can consider two profiles $\mathbf{R}, \mathbf{R}' \in \mathcal{R}_S^\infty$ such that

$$\begin{aligned} \mathbf{y}P(R'_t)\mathbf{z}P(R'_t)\mathbf{x} \quad & \forall t \in T; \\ R'_t|_{\{\mathbf{x},\mathbf{y}\}} = R_t|_{\{\mathbf{x},\mathbf{y}\}} \quad & \forall t \in \mathbb{N}. \end{aligned}$$

By the decisiveness of T for $\mathbf{y}$ over $\mathbf{z}$ (see (iv)) and for $\mathbf{z}$ over $\mathbf{x}$ (see (iii)) for f and transitivity, we obtain $\mathbf{y}P(f(\mathbf{R}'))\mathbf{x}$. By independence of irrelevant alternatives, $\mathbf{y}P(f(\mathbf{R}))\mathbf{x}$ and T is decisive for $\mathbf{y}$ over $\mathbf{x}$ for f. ■

Our version of Hansson's (1976) theorem is formulated for the selfish domain. Again, Pareto indifference is added so as to be able to apply Lemma 1.

Theorem 3 *If a social welfare function f satisfies selfish domain, weak Pareto, Pareto indifference and independence of irrelevant alternatives, then the set of all decisive coalitions for f is an ultrafilter on $\mathbb{N}$.*

Proof. Suppose f satisfies selfish domain, weak Pareto, Pareto indifference and independence of irrelevant alternatives. We need to show that the set of all decisive coalitions for f has the three properties of an ultrafilter.

1. If $\emptyset$ is decisive for f, we obtain $\mathbf{x}P(f(\mathbf{R}))\mathbf{y}$ and $\mathbf{y}P(f(\mathbf{R}))\mathbf{x}$ for any two alternatives $\mathbf{x}, \mathbf{y} \in X^\infty$ and for any profile $\mathbf{R} \in \mathcal{R}_S^\infty$ such that all generations are indifferent between $\mathbf{x}$ and $\mathbf{y}$, which is impossible. Thus, $\emptyset$ cannot be decisive for f.

2. Let $T \subseteq \mathbb{N}$. Let $\mathbf{x}, \mathbf{y}, \mathbf{z} \in X^\infty$ be such that x_t, y_t, z_t are pairwise distinct for all $t \in \mathbb{N}$. By selfish domain, we can define a profile $\mathbf{R} \in \mathcal{R}_S^\infty$ such that

$$\begin{aligned} &\mathbf{x}P(R_t)\mathbf{y} \text{ and } \mathbf{x}P(R_t)\mathbf{z} \qquad \forall t \in T; \\ &\mathbf{x}P(R_t)\mathbf{y} \text{ and } \mathbf{z}P(R_t)\mathbf{y} \qquad \forall t \in \mathbb{N} \setminus T. \end{aligned}$$

If $\mathbf{x}P(f(\mathbf{R}))\mathbf{z}$, T is decisive for $\mathbf{x}$ over $\mathbf{z}$ for f. Lemma 1 implies that T is decisive for f.

If $\neg\, \mathbf{x}P(f(\mathbf{R}))\mathbf{z}$, we have $\mathbf{z}f(\mathbf{R})\mathbf{x}$ by completeness. Furthermore, $\mathbf{x}P(f(\mathbf{R}))\mathbf{y}$ by weak Pareto. Transitivity implies $\mathbf{z}P(f(\mathbf{R}))\mathbf{y}$. Thus, $\mathbb{N} \setminus T$ is decisive for $\mathbf{y}$ over $\mathbf{z}$ and, by Lemma 1, $\mathbb{N} \setminus T$ is decisive for f.

3. Suppose T and T' are decisive for f. Let $\mathbf{x}, \mathbf{y}, \mathbf{z} \in X^\infty$ be such that x_t, y_t, z_t are pairwise distinct for all $t \in \mathbb{N}$. By selfish domain, we can define a profile $\mathbf{R} \in \mathcal{R}_S^\infty$ such that

$$\begin{aligned} \mathbf{x}P(R_t)\mathbf{y} \text{ and } \mathbf{x}P(R_t)\mathbf{z} &\qquad \forall t \in T \setminus T'; \\ \mathbf{z}P(R_t)\mathbf{x}P(R_t)\mathbf{y} &\qquad \forall t \in T \cap T'; \\ \mathbf{y}P(R_t)\mathbf{x} \text{ and } \mathbf{z}P(R_t)\mathbf{x} &\qquad \forall t \in T' \setminus T. \end{aligned}$$

Because T is decisive for f, we have $\mathbf{x}P(f(\mathbf{R}))\mathbf{y}$. Because T' is decisive for f, we have $\mathbf{z}P(f(\mathbf{R}))\mathbf{x}$. By transitivity, $\mathbf{z}P(f(\mathbf{R}))\mathbf{y}$. This implies that $T \cap T'$ is decisive for $\mathbf{z}$ over $\mathbf{y}$ for f. By Lemma 1, $T \cap T'$ is decisive for f. ■

The next step towards our characterization result consists of showing that Ferejohn and Page's (1978) dictatorship result is true on a selfish domain when Pareto indifference is added and multi-profile stationarity is used instead of stationarity.

Theorem 4 *If a social welfare function f satisfies selfish domain, weak Pareto, Pareto indifference, independence of irrelevant alternatives and multi-profile stationarity, then generation one is a dictator for f.*

Proof. Suppose f satisfies selfish domain, weak Pareto, Pareto indifference, independence of irrelevant alternatives and multi-profile stationarity. By Theorem 3, the set of decisive coalitions for f is an ultrafilter on $\mathbb{N}$. Suppose $\mathbb{N} \setminus \{1\}$ is decisive for f.

Let x and y be two distinct elements of the set X of per-period alternatives and let $\succeq$ be an ordering on X such that $xP(\succeq)y$. By selfish domain, we can define a profile $\mathbf{R} \in \mathcal{R}_S^\infty$ by letting, for all $t \in \mathbb{N}$ and for all $\mathbf{x}, \mathbf{y} \in X^\infty$,

$$\mathbf{x} R_t \mathbf{y} \;\Leftrightarrow\; x_t \succeq y_t.$$

Now consider the streams

$$\begin{aligned} \mathbf{x} &= (x, x, y, x, y, x, \ldots) = (x, \mathbf{y}); \\ \mathbf{y} &= (x, y, x, y, x, y, \ldots) = (x, \mathbf{z}); \\ \mathbf{z} &= (y, x, y, x, y, x, \ldots). \end{aligned}$$

Recall that $\mathbb{N} \setminus \{1\}$ is decisive for f.

If $\{2, 4, 6, \ldots\}$ is decisive for f, we have $\mathbf{x} P(f(\mathbf{R}))\mathbf{y}$. By multi-profile stationarity,

$$\mathbf{y} = \mathbf{x}_{\geq 2} P(f(\mathbf{R}_{\geq 2}))\mathbf{y}_{\geq 2} = \mathbf{z},$$

contradicting the decisiveness of $\{2, 4, 6, \ldots.\}$ for f.

If $\{2, 4, 6, \ldots\}$ is not decisive for f, property 2 of an ultrafilter implies that

$$\mathbb{N} \setminus \{2, 4, 6, \ldots\} = \{1, 3, 5, \ldots\}$$

is decisive for f. Because, in addition, $\mathbb{N} \setminus \{1\}$ is decisive for f, property 3 of an ultrafilter implies that

$$\{3, 5, 7, \ldots\} = \{1, 3, 5, \ldots\} \cap (\mathbb{N} \setminus \{1\})$$

is decisive for f. By virtue of the decisiveness of $\{3, 5, 7, \ldots\}$ for f, we have $\mathbf{y} P(f(\mathbf{R}))\mathbf{x}$. By multi-profile stationarity,

$$\mathbf{z} = \mathbf{y}_{\geq 2} P(f(\mathbf{R}_{\geq 2}))\mathbf{x}_{\geq 2} = \mathbf{y},$$

contradicting the decisiveness of $\{3, 5, 7, \ldots.\}$ for f.

Thus, in all cases, we obtain a contradiction to the assumption that $\mathbb{N} \setminus \{1\}$ is decisive for f. Therefore, because of property 2 of an ultrafilter, $\{1\}$ is decisive for f and, thus, generation one is a dictator for f. ■

The reason that generation one must be the generation that dictates is, again, a consequence of the unidirectional nature of the flow of time—and, thus, the unequal manner in which multi-profile stationarity treats generations. The example constructed in the above proof involving the streams $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}$ can be used to make this point: just as we lead the assumption that the coalition $\{2, 4, 6, \ldots\}$ is decisive for f to a contradiction in the relevant step of the proof, the same reasoning leads the assumption that generation two is a dictator to a contradiction. Clearly, the example is easily amended so as to apply to any generation other than generation one.

The final result of this paper characterizes f^{CD}.

Theorem 5 *A social welfare function f satisfies selfish domain, weak Pareto, Pareto indifference, independence of irrelevant alternatives and multi-profile stationarity if and only if $f = f^{CD}$.*

Proof. That f^{CD} satisfies the axioms can be verified by the reader. To prove the converse implication, suppose f satisfies the required axioms. It is sufficient to show that, for all $\mathbf{x}, \mathbf{y} \in X^\infty$ and for all $\mathbf{R} \in \mathcal{R}_S^\infty$,

$$\mathbf{x} I(f^{CD}(\mathbf{R}))\mathbf{y} \;\Rightarrow\; \mathbf{x} I(f(\mathbf{R}))\mathbf{y} \tag{1}$$

and

$$\mathbf{x} P(f^{CD}(\mathbf{R}))\mathbf{y} \;\Rightarrow\; \mathbf{x} P(f(\mathbf{R}))\mathbf{y}. \tag{2}$$

(1) follows immediately from Pareto indifference. To prove (2), suppose $t \in \mathbb{N}$, $\mathbf{x}, \mathbf{y} \in X^\infty$ and $\mathbf{R} \in \mathcal{R}_S^\infty$ are such that

$$x_\tau I(\succeq_\tau) y_\tau \;\forall \tau < t \;\text{ and }\; x_t P(\succeq_t) y_t.$$

If $t = 1$, let $\mathbf{z} = \mathbf{y}$; if $t \geq 2$, let $\mathbf{z} = (x_1, \ldots, x_{t-1}, \mathbf{y}_{\geq t})$. By Pareto indifference, $\mathbf{y} I(f(\mathbf{R}))\mathbf{z}$. Transitivity implies

$$\mathbf{x} f(\mathbf{R})\mathbf{y} \;\Leftrightarrow\; \mathbf{x} f(\mathbf{R})\mathbf{z}.$$

Together with the application of multi-profile stationarity $t-1$ times and noting that $\mathbf{z}_{\geq t} = \mathbf{y}_{\geq t}$, we obtain

$$\mathbf{x} f(\mathbf{R})\mathbf{y} \;\Leftrightarrow\; \mathbf{x} f(\mathbf{R})\mathbf{z} \;\Leftrightarrow\; \mathbf{x}_{\geq t} f(\mathbf{R}_{\geq t})\mathbf{z}_{\geq t} \;\Leftrightarrow\; \mathbf{x}_{\geq t} f(\mathbf{R}_{\geq t})\mathbf{y}_{\geq t}. \tag{3}$$

By Theorem 4, the relative ranking of $\mathbf{x}_{\geq t}$ and $\mathbf{y}_{\geq t}$ according to $\mathbf{R}_{\geq t}$ is determined by the strict preference for $\mathbf{x}$ over $\mathbf{y}$ according to the first generation in the profile $\mathbf{R}_{\geq t}$ (which is generation t in $\mathbf{R}$), so that $\mathbf{x}_{\geq t} P(f(\mathbf{R}_{\geq t}))\mathbf{y}_{\geq t}$ and, by (3), $\mathbf{x} P(f(\mathbf{R}))\mathbf{y}$. ■

5 Concluding Remarks

In concluding this paper, it may be worthwhile to clarify the relationship between the multi-profile version of intergenerational social choice theory analyzed in this paper, on the one hand, and the theory of evaluating infinite intergenerational utility streams, on the other. The latter theory capitalizes on the Koopmans (1960) analysis of impatience and the Diamond (1965) impossibility theorem on the existence of continuous social evaluation orderings on the set of infinite utility streams satisfying the Sidgwick (1907) anonymity principle and the Pareto principle. Among many contributions that appeared after Diamond (1965), those which are most relevant in the present context include Svensson (1980), Basu and Mitra (2003; 2007), Asheim, Mitra and Tungodden (2007), Bossert, Sprumont and Suzumura (2007) and Hara, Shinotsuka, Suzumura and Xu (2008). Although these two lines of inquiry are related in the sense that both are concerned with aggregating generational evaluations of their well-beings into an overall social evaluation, they contrast sharply in at least two respects. In the first place, the latter investigation is *welfaristic* in the sense of basing the overall social evaluation on the infinite-generational utility streams, whereas the former exercise is free from such an early commitment to this informational basis. In the second place, while the latter approach hinges squarely on the continuity assumption even in a vestigial form, the former has nothing to do with any continuity assumption on social evaluation orderings. More substantially, the Sidgwick (1907) anonymity principle, which plays a crucial role in establishing the Diamond impossibility theorem and related results, has nothing to do with our impossibility theorems. Since continuity is a requirement which is rather technical in nature, to get rid of the dependence on this assumption may be counted as a virtue rather than a vice. Although the Sidgwick anonymity principle has an obvious intuitive appeal, it is fortunate that we need not go against this plausible axiom in defending our approach. This principle can surely be added to the list of axioms but all that is thereby obtained is another set of Arrow-type impossibility results, some of which will even contain logical redundancies.

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