ON A UNIQUE NONDEGENERATE DISTRIBUTION OF AGENTS IN THE HUGGETT MODEL

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ABSTRACT

A theoretical curiosity remains in the Huggett [1993] model as to the possible existence of a unique *and* degenerate stationary distribution of agent types. This coincides with the possibility that an equilibrium individual state space may turn out to be trivial in the sense that every agent never escapes the binding common borrowing constraint. In this note, we extend and reinforce the proof of Lemma 3 in Huggett [1993]. By invoking a simple comparative-static argument, we establish that Huggett's result of a unique stationary equilibrium distribution of agents must be one that is nontrivial or nondegenerate.

KEYWORDS: Compactness; Individual state space; Stationary distribution
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1. INTRODUCTION

The seminal work of Huggett [1993] showed that there exists a unique stationary distribution of agent types, given by their individual states of asset and income endowment pairs. In the setting of Huggett [1993], the key insight on the risk-free rate anomaly arising from representative agent models, was obtained by an appeal to incomplete asset markets and precautionary saving motives. This framework was one of the key foundations for further quantitative research using heterogenous agent macroeconomics and is also part of the standard graduate curriculum [see e.g. Ljungqvist and Sargent, 2006]. In this class of models, important questions such as asset pricing puzzles [see e.g. Huggett, 1993; Aiyagari, 1994], and fiscal policy and taxation [see e.g. Heathcote, 2005], can now be seriously addressed.

Proving the existence of a unique stationary distribution of agent types in the model of Huggett [1993] is vital since the *stationary equilibrium* risk-free rate depends on this object. To establish this result, i.e. Theorem 2 in Huggett [1993], certain sufficiency conditions in theorem 2 of Hopenhayn and Prescott [1992] are required to be satisfied by the model. One of the requirements of the model is compactness of the agents' individual state space. Huggett [1993] showed the existence of a compact individual state space in any equilibrium where agents are behaving optimally. Intuitively, one needs to show that each agent indexed by an asset-endowment pair, $(a, e) \in S$, in making their optimal competitive decisions, would always remain in the set S every period.

However, the question remains open if this equilibrium individual state space S might turn out to be trivial, in the sense that every agent's common borrowing constraint binds forever. If so, the invariant probability measure of agent types will place all mass on this minimal credit level. We would like to point out that in establishing the result on the existence of an endogenously compact S by contradiction, Huggett [1993] omitted to consider that there might be two other valid contrary hypotheses, which leaves open the current theoretical curiosity.

In this note, we reinforce the proof of Lemma 3 in Huggett [1993] by showing that in fact, one can rule out one of these two contrary hypotheses toward the construction of the proof of Lemma 3, by invoking a simple comparative statics argument. We complete this missing check here. In other words, we establish that Huggett's result of a unique stationary equilibrium distribution of agents must be one that is nontrivial or nondegenerate. In practice, for plausible calibrations of the model, one does not encounter the problem of there being a trivial stationary equilibrium.[1] Our result serves to confirm the experience of numerical examples and to provide a general assurance for practitioners using numerical methods to solve models along the lines of Huggett [1993].

2. HUGGETT'S MODEL

In this section we first revisit the Huggett [1993] model. Then we provide a brief discussion on the notion of an endogenously compact individual state space and its implication for the existence of a unique stationary equilibrium distribution of agents.

In the Huggett model, time is discrete, and each period is indexed by $t \in \mathbb{N} := \{0, 1, ...\}$.[2] The population of agents has mass 1. Each measure zero agent receives a stream of stochastic endowment of consumption good. Let $E = \{e_l, e_h\}$, where $e_h > e_l$, be the set of endowment realizations. Each $(e_t)_{t\in\mathbb{N}}$ is governed by a given Markov chain (π, π_0) on E, where π is the stochastic matrix and π_0 the initial unconditional distribution on E. $\pi(e'|e) := \Pr\{e_{t+1} = e'|e_t = e\} > 0$, $e', e \in E$, is independent of t, and another agents' realization of e. Let $X = A \times E = [\underline{a}, +\infty) \times \{e_l, e_h\}$. The parameter $\underline{a}$ is interpreted as an exogenous borrowing constraint.

[1] A complete set of source codes and numerical solutions is available from the author on request.

[2] As is the usual convention, we may drop the explicit time-t subscript on variables, e.g. $x := x_t$ and $x' := x_{t+1}$.

2.1. **An individual's decision problem.** The *individual state* is $x := (a, e) \in X$. The individual takes as given the aggregate price $q > 0$. Suppose in an equilibrium there is a set $S := [\underline{a}, \overline{a}] \times \{e_l, e_h\} \subset X$ that generates Borel σ-algebra $\mathcal{B}(S)$. The dependence of the equilibrium q on the *aggregate state* given by a probability measure ψ on $(S, \mathcal{B}(S))$ is implicit.[3]

Each agent chooses consumption (c) and saving (s'). Let the agent's feasible action correspondence be $\Gamma(q) : A \times E \rightrightarrows \mathcal{B}(\mathbb{R}_+ \times A)$, where at each slice of Γ indexed by $(x; q)$, we have a description of the feasible choice set of an agent *currently* named x:[4]

$$\Gamma(x; q) = \left\{(c, a') : a + e \geq c + a'q, \;\; c \geq 0, \;\; a' \geq \underline{a}\right\}.$$

Denote $(x; q) \mapsto v(x; q) \in \mathbb{R}$ as an agent's value function. Each agent's Bellman equation is

$$v(x; q) = \max_{(c,a') \in \Gamma(x;q)} \left\{ u(c) + \beta \sum_{e' \in E} v(a', e'; q)\pi(e'|e) \right\}, \tag{1}$$

where $\beta \in (0, 1)$ and $u : \mathbb{R}_+ \to \mathbb{R}$ is strictly increasing, strictly concave, and twice continuously differentiable.

2.2. **Compact equilibrium individual state space $S \subset X$.** The notion of a stationary equilibrium is defined in Huggett [1993, p.956]. Given the Markov matrix for the endowment process, $\pi : E \to [0, 1]$, an initial individual state $x \in S$ for each agent, and an optimal decision rule, $(x; q) \mapsto \hat{a}(x; q)$, we can induce a time-invariant probability measure ψ on the measurable space $(S, \mathcal{B}(S))$ satisfying:

$$\psi(B) = \int_S P(x, B) \mathrm{d}\psi, \qquad \forall B \in \mathcal{B}(S),$$

where $P : S \times \mathcal{B}(S) \to [0, 1]$ is the equilibrium transition probability function.[5]

In Theorem 1, Huggett [1993] provides some sufficient conditions on the model such that given q, the solution to each agent's Bellman equation problem has some nice properties. Specifically, Theorem 1 in Huggett [1993] establishes that the optimal $\hat{a} : X \to [\underline{a}, \infty)$ is *continuous*, is either *strictly increasing* in a, if $a > \underline{a}$, or is *nondecreasing* in a if $a = \underline{a}$.

Theorem 1 and Lemmata 1-3 in Huggett [1993] tell us that each agent's optimal decision function for credit holdings, $\hat{a} : X \to [\underline{a}, \infty)$, must look something like that in Figure 1. In particular, this decision rule has the following properties:

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[3] In Huggett [1993], since the emphasis is on a notion of *recursive stationary equilibrium* where q is constant, we don't have to explicitly carry around the distribution of agent types ψ, as a relevant state variable. Instead, we only make the agents' problems dependent on q as a scalar parameter.

[4] For technical reasons, since A is a continuum, our agent's decision rules $c = c(a, e)$ and $a' = \hat{a}(a, e)$ need to be measurable functions belonging to $\Gamma(q)$. Hence we restrict such selections to only measurable subsets in the image of $\Gamma(q)$. These measurable subsets are in the Borel σ-algebra, $\mathcal{B}(\mathbb{R}_+ \times A)$ generated by $\mathbb{R}_+ \times A \ni (c, a')$.

[5] The details are discussed very nicely in Huggett [1993].

(1) When the current endowment is e_l, $\hat{a}(\cdot, e_l)$ is well below the 45°-line in (a, a')-space, if the borrowing constraint is not binding ($a > \underline{a}$). That is, a low endowment agent who is not currently credit constrained, accumulates credit below the current level (Lemma 1); and

(2) When the agent has high endowment, e_h, there is an asset level, $\overline{a}$, such that the policy function at e_h, $\hat{a}(\cdot, e_h)$, crosses the 45°-line in (a, a')-space. This is proved by Lemma 3, which uses both Lemma 1 and Lemma 2.

Thus, in an equilibrium, if there is to be an endogenous $\overline{a}$, as shown in Lemma 3 in Huggett [1993], which is the smallest fixed point satisfying $\hat{a}(a, e_h) = a$, then it is straightforward to deduce that $S := [\underline{a}, \overline{a}] \times \{e_l, e_h\}$ is an endogenously compact metric space. That is, each agent x beginning in S will always stay within S, or the equilibrium asset decision rule will be $\hat{a} : S \to [\underline{a}, \overline{a}]$.

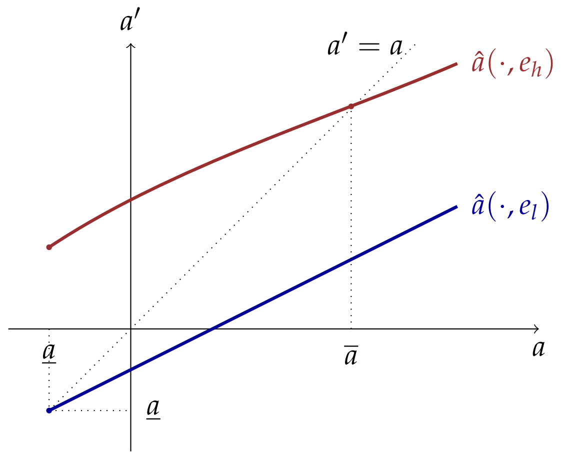

FIGURE 1. Characterization of optimal policy function $\hat{a} : X \to \mathbb{R}$. When $e = e_h$, for each a, $\hat{a}(a, e_h) > \hat{a}(a, e_l)$. If $\hat{a}(a, e) \geq \underline{a}$ binds, then $\hat{a}(\cdot, e)$ is nondecreasing in a.

Theorem 2 of Huggett, applying theorem 2 in Hopenhayn and Prescott [1992], provides sufficient conditions for the existence and uniqueness of a unique stationary distribution of agent types, ψ, for a given q. These conditions in turn include the requirement that S is a compact metric space.

3. A MISSING STEP

In this section, we complete the missing step required to ensure that indeed Huggett's endogenous upper bound $\overline{a}$ on assets is nontrivial. That is, we are required to show that $\underline{a} < \overline{a} < \infty$.

The aim (in Huggett's Lemma 3) is to show that there exists a fixed point $\overline{a}$ satisfying $\hat{a}(a, e_h) = a$ (and that it is a nontrivial fixed point: $\overline{a} > \underline{a}$). A contrary hypothesis to this would have three possible cases:

H1. $\hat{a}(a, e_h) < a$ for $a > \underline{a}$ and $\hat{a}(a, e_h) = a$ for $a = \underline{a}$,

H2. $\hat{a}(a, e_h) > a$ for $a > \underline{a}$ and $\hat{a}(a, e_h) = a$ for $a = \underline{a}$, and

H3. There is no a such that $\hat{a}(a, e_h) = a$.

These three hypothesis are depicted in Figure 2. In establishing the result on the existence of an endogenously compact S by contradiction in Lemma 3, Huggett [1993] made only the contrary hypothesis that there is no a such that $\hat{a}(a, e_h) = a$ (H3).

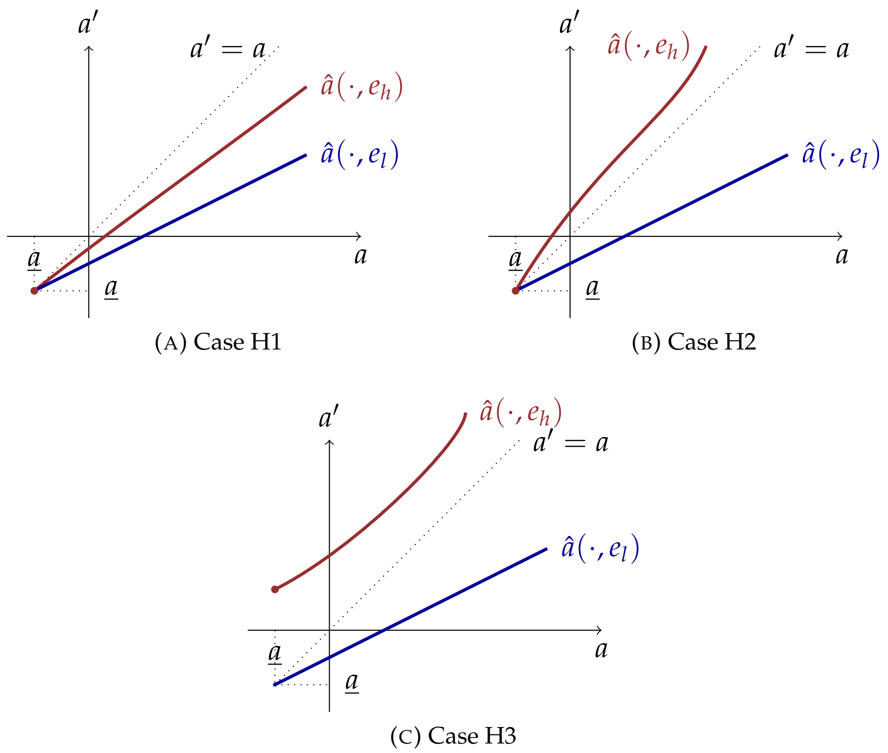

FIGURE 2. In proving Lemma 3 in Huggett [1993], suppose there is no a ($> \underline{a}$) such that $\hat{a}(a, e_h) = a$. *A priori* there may be three possible cases for the component function $\hat{a}(\cdot, e_h)$ that would satisfy this hypothesis. We can rule out cases H1 and H2.

It turns out, that only one of these contrary hypotheses is possible (i.e. H3), as was assumed in Huggett [1993]. However, it remains to be shown that this *must* be the only case, as we now show in the following lemma, which implies that $\hat{a}(a, e_h)$ must lie above $\underline{a}$ when $a = \underline{a}$. Thus, we ensure that we only need to make one contrary hypothesis (H3) to prove Huggett's Lemma 3.

Lemma. *The decision $\hat{a}(a, e)$ is strictly increasing in e for all $a \geq \underline{a}$.*

Proof. First, we show the case that a current individual state is $(\underline{a}, e)$. An optimal consumption decision $c(\underline{a}, e)$ for an agent currently named $(\underline{a}, e)$ must satisfy the first-order condition

$$u_c[c(\underline{a},e)] \geq \beta q^{-1}\mathbb{E}\left\{ u_c[c(\hat{a}(\underline{a},e),e')] \middle| e \right\}, \quad \text{with “=” if } \hat{a}(\underline{a},e) > \underline{a}. \tag{2}$$

Consider $(\underline{a},e) = (\underline{a},e_l)$. We then perturb e_l to $e_l + \Delta e =: e_h$. We want to show that $\hat{a}(\underline{a},e_h) > \hat{a}(\underline{a},e_l)$. By Theorem 1 in Huggett [1993], $\hat{a}(\underline{a},e_l) = \underline{a}$. Then (2) evaluated at $(\underline{a},e) = (\underline{a},e_l)$ is

$$u_c[c(\underline{a},e)] > \beta q^{-1}\mathbb{E}\left\{ u_c[c(\hat{a}(\underline{a},e),e')] \middle| e \right\}. \tag{3}$$

Suppose e_l increases to $e_l + \Delta e =: e_h$ and suppose $\hat{a}(\underline{a},e_l + \Delta e) = \hat{a}(\underline{a},e_l) = \underline{a}$. By Lemma 1 in Huggett [1993], this is consistent with the result $u_c[c(\underline{a},e_l)] = v_a(\underline{a},e_l) \geq v_a(\underline{a},e_h) = u_c[c(\underline{a},e_h)]$. So either the LHS of (3) declines or remain constant, and by strict concavity of u, $c(\underline{a},e_l) \leq c(\underline{a},e_h)$.

But since u is strictly concave, the agent would prefer to also shift some of the increase in e towards the next period, and across next-period states. That is, in the RHS of (3), for each fixed $e' \in E$, $u_c[c(\hat{a}(\underline{a},e),e')]$ must fall. From the agent's budget constraint, this implies that $\hat{a}(\underline{a},e)$ must increase in e. That is, if $e_h > e_l$, then $\hat{a}(\underline{a},e_h) > \hat{a}(\underline{a},e_l) = \underline{a}$, so then (2) would hold with equality at $(\underline{a},e_h)$.

Second, consider $a > \underline{a}$. By Theorem 1 in Huggett [1993], $\hat{a}(\cdot,e)$ is strictly increasing and continuous in $a > \underline{a}$. By Lemma 1 in Huggett [1993], $\hat{a}(a,e_l) < a$ for $a > \underline{a}$. Using these facts, and since we have shown $\hat{a}(\underline{a},e_h) > \hat{a}(\underline{a},e_l) = \underline{a}$, then there exists some a such that $\hat{a}(a,e_h) > a > \hat{a}(a,e_l) \geq \underline{a}$, and for all a, $\hat{a}(a,e_h) > \hat{a}(a,e_l) \geq \underline{a}$. □

4. Discussion

The hypotheses H1 (Figure 2.A) and H2 (Figure 2.B) can thus be ruled out since $\hat{a}(\cdot,e_h)$ must be above the 45°-line in (a,a')-space at the point $\underline{a}$. By applying our Lemma, the proof of Huggett's Lemma 3 is then complete, where we would have ruled out any possible trivial equilibrium individual state space as well. The idea is that now, given the Lemma above, we can just assume one case – that $\hat{a}(a,e_h) > a$ for all a, so that there is no fixed point for $\hat{a}(a,e_h)$ in (a,a')-space, but then arrive at a contradiction. The conclusion would have to be that there is an $a^* > \underline{a}$ that is the fixed point, and we can take the least fixed point to be $a^* = \bar{a}$, the endogenous upper bound on assets. Finally, given these results, Theorem 2 of Huggett follows to establish existence and uniqueness of a stationary distribution of agent types, which is now guaranteed to be nondegenerate.

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