Consistent Rationalizability*

Walter Bossert
Département de Sciences Économiques and CIREQ
Université de Montréal
C.P. 6128, succursale Centre-ville
Montréal QC H3C 3J7
Canada
FAX: (+1 514) 343 7221
e-mail: Walter.Bossert@umontreal.ca

Yves Sprumont
Département de Sciences Économiques and CIREQ
Université de Montréal
C.P. 6128, succursale Centre-ville
Montréal QC H3C 3J7
Canada
FAX: (+1 514) 343 7221
e-mail: Yves.Sprumont@umontreal.ca

Kotaro Suzumura
Institute of Economic Research
Hitotsubashi University
Kunitachi, Tokyo 186-8603
Japan
FAX: (+81 42) 580 8353
e-mail: suzumura@ier.hit-u.ac.jp

June 2002

* Financial support through grants from the Social Sciences and Humanities Research Council of Canada, the Fonds pour la Formation de Chercheurs et l'Aide à la Recherche of Québec, and a Grant-in-Aid for Scientific Research for Priority Areas (B) Number 603 from the Ministry of Education, Science and Culture of Japan is gratefully acknowledged.

Abstract

Consistency of a binary relation requires any preference cycle to involve indifference only. As shown by Suzumura (1976b), consistency is necessary and sufficient for the existence of an ordering extension of a relation. Because of this important role of consistency, it is of interest to examine the rationalizability of choice functions by means of consistent relations. We describe the logical relationships between the different notions of rationalizability obtained if reflexivity or completeness are added to consistency, both for greatest-element rationalizability and for maximal-element rationalizability. All but one notion of consistent rationalizability are characterized for general domains, and all of them are characterized for domains that contain all two-element subsets of the universal set. *Journal of Economic Literature* Classification No.: D11.

Keywords: Rational Choice, Consistency, Binary Domains.

1 Introduction

Samuelson (1938) began his seminal paper on revealed preference theory with a remark that "[f]rom its very beginning the theory of consumer's choice has marched steadily towards greater generality, sloughing off at successive stages unnecessarily restrictive conditions" (Samuelson, 1938, p. 61). Even after Samuelson (1938; 1947, Chapter V; 1948; 1950) laid the foundations of "the theory of consumer's behaviour freed from any vestigal traces of the utility concept" (Samuelson, 1938, p. 71), the exercise of Ockham's razor persisted within revealed preference theory. Capitalizing on Georgescu-Roegen's (1954, p. 125; 1966, p. 222) observation that the intuitive justification of the axioms of revealed preference theory has nothing to do with the special form of budget sets but, instead, is based on the implicit consideration of choices from two-element sets, Arrow (1959) expanded the analysis of rational choice and revealed preference beyond consumer choice problems. He pointed out that "the demand-function point of view would be greatly simplified if the range over which the choice functions are considered to be determined is broadened to include all finite sets" (Arrow, 1959, p. 122). Sen (1971, p. 312) defended Arrow's domain assumption by posing two important questions: "why assume the axioms [of revealed preference] to be true only for 'budget sets' and not for others?" and "[a]re there reasons to expect that some of the rationality axioms will tend to be satisfied in choices over 'budget sets' but not for other choices?"

While it is certainly desirable to liberate revealed preference theory from the narrow confinement of budget sets, the admission of all finite subsets of the universal sets into the domain of a choice function may well be unsuitable for many applications. In this context, two important groups of contributions stand out. In the first place, Richter (1966; 1971), Hansson (1968) and Suzumura (1976a; 1977; 1983, Chapter 2) developed the theory of rational choice and revealed preference for choice functions with general nonempty domains which do not impose any extraneous restrictions whatsoever on the class of feasible sets. In the second place, Sen (1971) showed that Arrow's results (as well as others with similar features) do not hinge on the full power of the assumption that *all* finite sets are included in the domain of a choice function—it suffices if the domain contains all two-element and three-element sets.

It was in view of this current state of the art that Bossert, Sprumont and Suzumura (2001) examined two crucial types of domains in an analysis of several open questions in the theory of rational choice. The first is the general domain à la Richter, Hansson and Suzumura, and the second is the class of *base domains* which include all singletons and all two-element subsets of the universal set. The status of the general domain seems to be impeccable, as the theory developed on this domain is relevant in whatever choice situations we may care to specify. The base domains also seem to be on safe ground, as the concept of rational choice as maximizing choice is intrinsically connected with pairwise comparisons: singletons can be viewed as pairs with identical components, whereas two-element sets represent pairs of distinct alternatives. As Arrow (1951, p. 16; 1963) put it, "one of the consequences of the assumptions of rational choice is that the choice in any environment can be determined by a knowledge of the choices in two-element environments."

In this paper, we focus on the rationalizability of choice functions by means of *consistent* relations. The concept of consistency was first introduced by Suzumura (1976b), and it is a weakening of transitivity requiring that any revealed preference cycle should involve indifference only. As was shown by Suzumura (1976b; 1983, Chapter 1), consistency is necessary and sufficient for the existence of an ordering extension of a binary relation. For that reason, consistency is a central property for the analysis of rational choice as well: in order to obtain a rationalizing relation that is an ordering, an extension procedure is, in general, required in order to ensure that the rationalization is complete. We examine consistent rationalizability under two domain assumptions. The first is, again, the general domain assumption where no restrictions whatsoever are imposed, and the second weakens the base domain hypothesis: we merely require the domain to contain all two-element sets but not necessarily all singletons, and we refer to those domains as *binary domains.* Thus, our results are applicable in a wide range of choice problems. It is worth pointing out that we do not require the triples to be part of our domain. In that sense, our approach is more general than Sen's.

Depending on the additional properties that can be imposed on rationalizations (reflexivity and completeness), different notions of consistent rationalizability can be defined. We characterize all but one of those notions in the general case, and all of them in the case of binary domains. It is remarkable that we obtain full characterization results on binary domains (in particular, on domains that do not have to contain any triples), even though consistency imposes a restriction on possible cycles of any length.

In Section 2, the notation and our basic definitions are presented, along with some preliminary lemmas. Section 3 develops the theory of consistent rationalizability on general domains, whereas Section 4 expounds the corresponding theory on binary domains. Some concluding remarks are collected in Section 5.

2 Preliminaries

The set of positive (nonnegative) integers is denoted by $\mathbb{N}$ ($\mathbb{N}_0$). For a set S, $|S|$ is the cardinality of S. Let X be a universal nonempty set of alternatives. $\mathcal{X}$ is the power set of X excluding the empty set. A choice function is a mapping $C: \Sigma \to \mathcal{X}$ such that $C(S) \subseteq S$ for all $S \in \Sigma$, where $\Sigma \subseteq \mathcal{X}$ with $\Sigma \neq \emptyset$ is the domain of C. Note that C maps Σ into the set of all *nonempty* subsets of X. Thus, using Richter's (1971) terminology, the choice function C is assumed to be *decisive.* Let $C(\Sigma)$ denote the image of Σ under C, that is, $C(\Sigma) = \cup_{S \in \Sigma} C(S)$. In addition to arbitrary nonempty domains, to be called *general domains,* we consider *binary domains* which are domains $\Sigma \subseteq \mathcal{X}$ such that $\{S \in \mathcal{X} \mid |S| = 2\} \subseteq \Sigma$.

Let $R \subseteq X \times X$ be a (binary) relation on X. The asymmetric factor $P(R)$ of R is given by $(x, y) \in P(R)$ if and only if $(x, y) \in R$ and $(y, x) \notin R$ for all $x, y \in X$. The symmetric factor $I(R)$ of R is defined by $(x, y) \in I(R)$ if and only if $(x, y) \in R$ and $(y, x) \in R$ for all $x, y \in X$. The noncomparable factor $N(R)$ of R is given by $(x, y) \in N(R)$ if and only if $(x, y) \notin R$ and $(y, x) \notin R$ for all $x, y \in X$.

A relation $R \subseteq X \times X$ is (i) reflexive if, for all $x \in X$, $(x, x) \in R$; (ii) complete if, for all $x, y \in X$ such that $x \neq y$, $(x, y) \in R$ or $(y, x) \in R$; (iii) transitive if, for all $x, y, z \in X$, $[(x, y) \in R$ and $(y, z) \in R]$ implies $(x, z) \in R$; (iv) consistent if, for all $K \in \mathbb{N} \setminus \{1\}$ and for all $x^0, \ldots, x^K \in X$, $(x^{k-1}, x^k) \in R$ for all $k \in \{1, \ldots, K\}$ implies $(x^K, x^0) \notin P(R)$; (v) P-acyclical if, for all $K \in \mathbb{N} \setminus \{1\}$ and for all $x^0, \ldots, x^K \in X$, $(x^{k-1}, x^k) \in P(R)$ for all $k \in \{1, \ldots, K\}$ implies $(x^K, x^0) \notin P(R)$.

The transitive closure of $R \subseteq X \times X$ is denoted by $\overline{R}$, that is, for all $x, y \in X$, $(x, y) \in \overline{R}$ if there exist $K \in \mathbb{N}$ and $x^0, \ldots, x^K \in X$ such that $x = x^0$, $(x^{k-1}, x^k) \in R$ for all $k \in \{1, \ldots, K\}$ and $x^K = y$. Clearly, $\overline{R}$ is transitive and, because we can set $K = 1$, it follows that $R \subseteq \overline{R}$.

The *direct revealed preference relation* $R_C \subseteq X \times X$ of a choice function C with an arbitrary domain Σ is defined as follows. For all $x, y \in X$, $(x, y) \in R_C$ if there exists $S \in \Sigma$ such that $x \in C(S)$ and $y \in S$. The *(indirect) revealed preference relation* of C is the transitive closure $\overline{R_C}$ of the direct revealed preference relation R_C.

For $S \in \Sigma$ and a relation $R \subseteq X \times X$, the set of R-greatest elements in S is $\{x \in S \mid (x, y) \in R \text{ for all } y \in S\}$, and the set of R-maximal elements in S is $\{x \in S \mid (y, x) \notin P(R) \text{ for all } y \in S\}$. A choice function C is greatest-element rationalizable if there exists a relation R on X, to be called a *G-rationalization,* such that $C(S)$ is equal to the set of R-greatest elements in S for all $S \in \Sigma$. C is maximal-element rationalizable if there exists a relation R on X, to be called an *M-rationalization,* such that $C(S)$ is equal to the set of R-maximal elements in S for all $S \in \Sigma$. We use the term *rationalization* in general discussions where it is not specified whether greatest-element rationalizability or maximal-element rationalizability is considered.

Depending on the properties that we might want to impose on a rationalization, different notions of rationalizability can be defined. For simplicity of presentation, we use the following notation. **G** (respectively **RG**; **CG**; **RCG**) stands for greatest-element rationalizability by means of a consistent (respectively reflexive and consistent; complete and consistent; reflexive, complete and consistent) G-rationalization. Analogously, **M** (respectively **RM**; **CM**; **RCM**) is maximal-element rationalizability by means of a consistent (respectively reflexive and consistent; complete and consistent; reflexive, complete and consistent) M-rationalization. Note that we do not identify consistency explicitly in these acronyms even though it is assumed to be satisfied by the rationalization in question. This is because consistency is required in all of the theorems presented in this paper and the use of another piece of notation would be redundant and likely increase the complexity of our exposition. However, note that the two lemmas stated below do not require consistency. In particular, part (i) of Lemma 2 ceases to be true if consistency is added as a requirement; see also Theorem 1.

We conclude this section with two preliminary results. We first present the following lemma, the first part of which is due to Samuelson (1938; 1948); see also Richter (1971). It states that the direct revealed preference relation must be contained in any G-rationalization and, moreover, that if an alternative x is directly revealed preferred to an alternative y, then y cannot be strictly preferred to x by any M-rationalization.

Lemma 1 *(i) If R is a G-rationalization of C, then $R_C \subseteq R$.*

(ii) If R is an M-rationalization of C, then $R_C \subseteq R \cup N(R)$.

Proof. (i) Suppose that R is a G-rationalization of C and $x, y \in X$ are such that $(x, y) \in R_C$. By definition of R_C, there exists $S \in \Sigma$ such that $x \in C(S)$ and $y \in S$. Because R is a G-rationalization of C, we obtain $(x, y) \in R$.

(ii) Suppose R is an M-rationalization of C and $x, y \in X$ are such that $(x, y) \in R_C$. By way of contradiction, suppose $(x, y) \notin R \cup N(R)$. Therefore, $(y, x) \in P(R)$. Because R maximal-element rationalizes C, this implies $x \notin C(S)$ for all $S \in \Sigma$ such that $y \in S$. But this contradicts the hypothesis $(x, y) \in R_C$. ■

Our second preliminary observation concerns the relationship between maximal-element rationalizability and greatest-element rationalizability when no further restrictions are imposed on a rationalization. This applies, in particular, when consistency is not imposed. Moreover, an axiom which is necessary for either form of rationalizability is presented. This requirement is referred to as the V-axiom in Richter (1971); we call it direct-revelation coherence in order to have a systematic terminology throughout this paper.

Direct-Revelation Coherence: For all $S \in \Sigma$, for all $x \in S$, if $(x, y) \in R_C$ for all $y \in S$, then $x \in C(S)$.

Suzumura (1976a) establishes that, in the absence of any requirements on a rationalization, maximal-element rationalizability implies greatest-element rationalizability. Furthermore, Richter (1971) shows that direct-revelation coherence is necessary for greatest-element rationalizability by an arbitrary G-rationalization on an arbitrary domain. We summarize these observations in the following lemma. For completeness, we provide a proof.

Lemma 2 *(i) If C is maximal-element rationalizable, then C is greatest-element rationalizable.*

(ii) If C is greatest-element rationalizable, then C satisfies direct-revelation coherence.

Proof. (i) Suppose R is an M-rationalization of C. It is straightforward to verify that $R' = \{(x, y) \mid (y, x) \notin P(R)\}$ is a G-rationalization of C.

(ii) Suppose R is a G-rationalization of C, and let $S \in \Sigma$ and $x \in S$ be such that $(x, y) \in R_C$ for all $y \in S$. By part (i) of Lemma 1, $(x, y) \in R$ for all $y \in S$. Because R is a G-rationalization of C, this implies $x \in C(S)$. ■

Richter (1971) shows that direct-revelation coherence is not only necessary but also sufficient for greatest-element rationalizability on an arbitrary domain, without any further restrictions imposed on the G-rationalization. Moreover, the axiom is necessary and sufficient for greatest-element rationalizability by a reflexive (but otherwise unrestricted) rationalization on an arbitrary domain. The requirement remains, of course, necessary for greatest-element rationalizability if we restrict attention to binary domains. As shown below, if we add consistency as a requirement on a rationalization, direct-revelation coherence by itself is sufficient for neither greatest-element rationalizability nor for maximal-element rationalizability, even on binary domains.

3 General Domains

In this section, we impose no restrictions on the domain Σ. We begin our analysis by providing a full description of the logical relationships between the different notions of rationalizability that can be defined, given our consistency assumption imposed on a rationalization. The possible definitions of rationalizability that can be obtained depend on whether reflexivity or completeness are added to consistency. Furthermore, a distinction between greatest-element rationalizability and maximal-element rationalizability is made. For convenience, a diagrammatic representation is employed: all axioms that are depicted within the same box are equivalent, and an arrow pointing from one box b to another box b' indicates that the axioms in b imply those in b', and the converse implication is not true without further assumptions regarding the domain of C.

Theorem 1 *Suppose Σ is a general domain. Then*

RCG, **CG**, **RCM**, **CM**	
↓	↓
RG, **G**	**RM**, **M**

Proof. We proceed as follows. In Step 1, we prove the equivalence of all axioms that appear in the same box. In Step 2, we show that all implications depicted in the theorem statement are valid. In Step 3, we provide examples demonstrating that no further implications are true in general.

Step 1. For each of the three boxes, we show that all axioms listed in the box are equivalent.

1.a. We first prove the equivalence of the axioms in the top box.

Clearly, **RCG** implies **CG** and **RCM** implies **CM**. Moreover, if a relation R is reflexive and complete, it follows that the set of R-greatest elements in S is equal to the set of R-maximal elements in S for any $S \in \Sigma$. Therefore, **RCG** and **RCM** are equivalent.

To see that **CM** implies **RCM**, suppose R is a consistent and complete M-rationalization of C. Let

$$R' = R \cup \{(x, x) \mid x \in X\}.$$

Clearly, R' is reflexive. R' is consistent and complete because R is. That R' is an M-rationalization of C follows immediately from the observation that R is.

To complete Step 1.a of the proof, it is sufficient to show that **CG** implies **RCG**. Suppose R is a consistent and complete G-rationalization of C. Let

$$\begin{aligned} R' &= (R \cup \{(x,x) \mid x \in X\} \cup \{(y,x) \mid x \notin C(\Sigma) \text{ and } y \in C(\Sigma)\}) \\ &\setminus \{(x,y) \mid x \notin C(\Sigma) \text{ and } y \in C(\Sigma)\}. \end{aligned}$$

Clearly, R' is reflexive by definition.

To show that R' is complete, let $x, y \in X$ be such that $x \neq y$ and $(x,y) \notin R'$. By definition of R', this implies

(i) $(x,y) \notin R$ and $[x \notin C(\Sigma)$ or $y \in C(\Sigma)]$

or

(ii) $x \notin C(\Sigma)$ and $y \in C(\Sigma)$.

In case (i), the completeness of R implies $(y,x) \in R$ and, by definition of R', we obtain $(y,x) \in R'$. In case (ii), $(y,x) \in R'$ follows immediately from the definition of R'.

Next, we show that R' is consistent. Let $K \in \mathbb{N} \setminus \{1\}$ and $x^0, \ldots, x^K \in X$ be such that $(x^{k-1}, x^k) \in R'$ for all $k \in \{1, \ldots, K\}$. Clearly, we can, without loss of generality, assume that $x^{k-1} \neq x^k$ for all $k \in \{1, \ldots, K\}$. We distinguish two cases.

(i) $x^0 \notin C(\Sigma)$. In this case, it follows that $x^1 \notin C(\Sigma)$; otherwise we would have $(x^1, x^0) \in P(R')$ by definition of R', contradicting our hypothesis. Successively applying this argument to all $k \in \{1, \ldots, K\}$, we obtain $x^k \notin C(\Sigma)$ for all $k \in \{1, \ldots, K\}$. By definition of R', this implies $(x^{k-1}, x^k) \in R$ for all $k \in \{1, \ldots, K\}$. By the consistency of R, we must have $(x^K, x^0) \notin P(R)$. Because $x^K \notin C(\Sigma)$, this implies, according to the definition of R', $(x^K, x^0) \notin P(R')$.

(ii) $x^0 \in C(\Sigma)$. If $x^K \notin C(\Sigma)$, $(x^0, x^K) \in R'$ follows immediately from the definition of R'. If $x^K \in C(\Sigma)$, it follows that $x^{K-1} \in C(\Sigma)$; otherwise we would have $(x^K, x^{K-1}) \in P(R')$ by definition of R', contradicting our hypothesis. Successively applying this argument to all $k \in \{1, \ldots, K\}$, we obtain $x^k \in C(\Sigma)$ for all $k \in \{1, \ldots, K\}$. By definition of R', this implies $(x^{k-1}, x^k) \in R$ for all $k \in \{1, \ldots, K\}$. By the consistency of R, we must have $(x^K, x^0) \notin P(R)$. Because $x^0 \in C(\Sigma)$, this implies, according to the definition of R', $(x^K, x^0) \notin P(R')$.

Finally, we show that R' is a G-rationalization of C. Let $S \in \Sigma$ and $x \in S$.

Suppose first that

$$(x,y) \in R' \text{ for all } y \in S. \tag{1}$$

We show that this implies $(x, y) \in R$ for all $y \in S$ which, in turn, implies $x \in C(S)$ because R is a G-rationalization of C. By way of contradiction, suppose there exists $y \in S$ such that

$$(x, y) \notin R. \tag{2}$$

Then $x \notin C(S)$ because R is a G-rationalization of C. The nonemptiness of $C(S)$ implies that there exists $z \in S \setminus \{x\}$ such that $z \in C(S)$ and, because $C(S) \subseteq C(\Sigma)$, we obtain

$$z \in C(\Sigma). \tag{3}$$

Because $z \in S$, we have

$$(x, z) \in R' \tag{4}$$

by (1). If $x = y$, (2) implies $(x, x) \notin R$ and, because R is a G-rationalization of C, $x \notin C(\Sigma)$. By (3) and the definition of R', we obtain $(z, x) \in P(R')$, contradicting (4). Similarly, if $(x, z) \notin R$, (3) and the definition of R' together imply $(x, z) \notin R'$, again contradicting (4). Therefore, we must have $x \neq y$ and $(x, z) \in R$. Because R is a G-rationalization of C, $z \in C(S)$ and $y \in S$ together imply $(z, y) \in R$. By the consistency of R, we obtain $(y, x) \notin P(R)$. Because $x \neq y$, the completeness of R implies $(x, y) \in R$, contradicting (2).

To prove the converse implication, suppose $x \in C(S)$. Because R is a G-rationalization of C, we have $(x, y) \in R$ for all $y \in S$. In particular, this implies $(x, x) \in R$ and, according to the definition of R', we obtain $(x, y) \in R'$ for all $y \in S$.

1.b. The proof that **RM** and **M** are equivalent is analogous to the proof of the equivalence of **RCM** and **CM** in Step 1.a.

1.c. Clearly, **RG** implies **G**. Conversely, suppose R is a consistent G-rationalization of C. Let

$$R' = (R \cup \{(x, x) \mid x \in X\}) \setminus \{(x, y) \mid x \notin C(\Sigma) \text{ and } x \neq y\}.$$

Clearly, R' is reflexive.

Next, we prove that R' is consistent. Suppose, by way of contradiction, that there exist $K \in \mathbb{N} \setminus \{1\}$ and $x^0, \ldots, x^K \in X$ such that $(x^{k-1}, x^k) \in R'$ for all $k \in \{1, \ldots, K\}$ and $(x^K, x^0) \in P(R')$. Again, we can assume that $x^{k-1} \neq x^k$ for all $k \in \{1, \ldots, K\}$. Because of that, the definition of R' implies that $(x^{k-1}, x^k) \in R$ for all $k \in \{1, \ldots, K\}$ and $(x^K, x^0) \in R$. Because $(x^K, x^0) \in P(R')$, it follows that $(x^0, x^K) \notin R'$. If $(x^0, x^K) \in R$, the definition of R' implies that $x^0 \notin C(\Sigma)$. But then, by definition of R', we have $(x^0, x^1) \notin R'$, a contradiction. Therefore, $(x^0, x^K) \notin R$ and hence $(x^K, x^0) \in P(R)$, contradicting the consistency of R.

It remains to be shown that R' is a G-rationalization of C. Let $S \in \Sigma$ and $x \in S$.

First, suppose $(x, y) \in R'$ for all $y \in S$. If $S = \{x\}$, $x \in C(S)$ follows from the nonemptiness of $C(S)$. If $S \neq \{x\}$,

$$(x, y) \in R \text{ for all } y \in S \setminus \{x\} \tag{5}$$

by definition of R'. Furthermore, $(x, y) \in R'$ for all $y \in S$ and the definition of R' together imply $x \in C(\Sigma)$. Therefore, because R is a G-rationalization of C, we must have $(x, x) \in R$. Together with (5), it follows that $(x, y) \in R$ for all $y \in S$ and, because R is a G-rationalization of C, $x \in C(S)$.

Finally, suppose $x \in C(S)$. This implies $(x, y) \in R$ for all $y \in S$ because R is a G-rationalization of C. Furthermore, because $C(S) \subseteq C(\Sigma)$, we have $x \in C(\Sigma)$. By definition of R', this implies $(x, y) \in R'$ for all $y \in S$.

Step 2. The strict implications depicted by the arrows in the theorem statement are straightforward.

Step 3. Given Steps 1 and 2, to prove that no further implications are valid, it is sufficient to provide examples showing that (a) **M** does not imply **G**; and (b) **G** does not imply **M**. Note that this independence of **M** and **G** in the presence of consistency does not contradict part (i) of Lemma 2—consistency is not required in the lemma.

3.a. **M** does not imply **G**.

Example 1 *Let* $X = \{x, y, z\}$ *and* $\Sigma = \{\{x, y\}, \{x, z\}, \{y, z\}\}$. *For future reference, note that* Σ *is a binary domain. Define the choice function* C *by letting* $C(\{x, y\}) = \{x, y\}$, $C(\{x, z\}) = \{x, z\}$ *and* $C(\{y, z\}) = \{y\}$. *This choice function is maximal-element rationalizable by the consistent (and reflexive) rationalization*

$$R = \{(x, x), (y, y), (y, z), (z, z)\}.$$

Suppose C *is greatest-element rationalizable by a consistent rationalization* R'. *Because* $C(\Sigma) = X$, *greatest-element rationalizability implies that* R' *is reflexive. Therefore, because* $y \in C(\{y, z\})$ *and* $z \notin C(\{y, z\})$, *we must have* $(y, z) \in R'$ *and* $(z, y) \notin R'$. *Therefore,* $(y, z) \in P(R')$. *Because* R' *is a G-rationalization of* C, $z \in C(\{x, z\})$ *implies* $(z, x) \in R'$ *and* $x \in C(\{x, y\})$ *implies* $(x, y) \in R'$. *This yields a contradiction to the assumption that* R' *is consistent.*

3.b. To prove that **G** does not imply **M**, we employ an example due to Suzumura (1976a, pp. 151–152).

Example 2 *Let* $X = \{x, y, z\}$ *and* $\Sigma = \{\{x, y\}, \{x, z\}, \{x, y, z\}\}$*, and define* $C(\{x, y\}) = \{x, y\}$*,* $C(\{x, z\}) = \{x, z\}$ *and* $C(\{x, y, z\}) = \{x\}$*. This choice function is greatest-element rationalizable by the consistent (and reflexive) rationalization*

$$R = \{(x, x), (x, y), (x, z), (y, x), (y, y), (z, x), (z, z)\}.$$

Suppose R' *is an M-rationalization of* C*. Because* $z \in C(\{x, z\})$*, maximal-element rationalizability implies* $(x, z) \notin P(R')$ *and, consequently,* $z \notin C(\{x, y, z\})$ *implies* $(y, z) \in P(R')$*. Analogously,* $y \in C(\{x, y\})$ *implies, together with maximal-element rationalizability,* $(x, y) \notin P(R')$ *and, consequently,* $y \notin C(\{x, y, z\})$ *implies* $(z, y) \in P(R')$*. But this contradicts the above observation that we must have* $(y, z) \in P(R')$*. Note that consistency (or any other property) of* R' *is not invoked in the above argument. Moreover,* R *is reflexive. Thus,* **RG** *does not even imply maximal-element rationalizability by an arbitrary rationalization.*

■

We now provide characterizations of two of the three notions of rationalizability identified in the above theorem. The first is a straightforward consequence of Richter's (1966) result and the observation that consistency is equivalent to transitivity in the presence of reflexivity and completeness. Richter (1966) shows that the *congruence* axiom is necessary and sufficient for greatest-element rationalizability by a transitive, reflexive and complete rationalization. Congruence is defined as follows.

Congruence: For all $x, y \in X$, for all $S \in \Sigma$, if $(x, y) \in \overline{R_C}$, $y \in C(S)$ and $x \in S$, then $x \in C(S)$.

We obtain

Theorem 2 *C satisfies* **RCG** *if and only if* C *satisfies congruence.*

Proof. As is straightforward to verify, a relation is consistent, reflexive and complete if and only if it is transitive, reflexive and complete. The result now follows immediately from the equivalence of congruence and greatest-element rationalizability by a transitive, reflexive and complete rationalization established by Richter (1966). ■

In order to characterize **G** (and, therefore, **RG**; see Theorem 1), we define the *consistent closure* of the direct revealed preference relation R_C. The consistent closure of a relation is analogous to the transitive closure in the sense that the idea is to add all pairs to the relation R_C that must be in a G-rationalizing relation due to the requirement that the rationalization be consistent. In contrast to the transitive closure, however, this addition of pairs to R_C cannot be performed in a single step. After we add all pairs of alternatives to R_C that need to be added as a consequence of consistency, the resulting relation may require further additions that are not imposed in the first step. Consequently, we proceed in a recursive fashion. We start with the direct revealed preference relation R_C, that is, we let $R_C^0 = R_C$. Now suppose R_C^t has been defined for $t \in \mathbb{N}_0$, and define

$$R_C^{t+1} = R_C^t \cup \{(x, y) \mid (x, y) \in \overline{R_C^t} \text{ and } (y, x) \in R_C^t\}.$$

Because X is not necessarily finite, new pairs may be added at each step. However, even if X is not countable, a countable number of steps turns out to suffice in order to capture all additions that are relevant for our purposes. The consistent closure of R_C is now obtained as

$$R_C^* = \cup_{t \in \mathbb{N}_0} R_C^t.$$

Clearly, for all $t \in \mathbb{N}_0$,

$$R_C^t \subseteq R_C^{t+1} \subseteq R_C^*. \tag{6}$$

The following axiom is a strengthening of direct-revelation coherence which we call consistent-closure coherence. It is obtained by replacing R_C with R_C^* in the definition of direct-revelation coherence.

Consistent-Closure Coherence: For all $S \in \Sigma$, for all $x \in S$, if $(x, y) \in R_C^*$ for all $y \in S$, then $x \in C(S)$.

Analogously to Lemma 1, we obtain

Lemma 3 *(i) If R is a consistent G-rationalization of C, then $R_C^* \subseteq R$.*
(ii) If R is a consistent M-rationalization of C, then $R_C^* \subseteq R \cup N(R)$.

Proof. (i) Suppose R is a consistent G-rationalization of C.

We first prove by induction that $R_C^t \subseteq R$ for all $t \in \mathbb{N}_0$. By Lemma 1, $R_C^0 = R_C \subseteq R$. Now suppose $R_C^t \subseteq R$ for some $t \in \mathbb{N}_0$. Let $(x, y) \in R_C^{t+1}$. By definition, this implies $(x, y) \in R_C^t$ or $[(x, y) \in \overline{R_C^t}$ and $(y, x) \in R_C^t]$. If $(x, y) \in R_C^t$, $(x, y) \in R$ follows from the induction hypothesis. Now suppose the second possibility applies. Again using the induction hypothesis, $(y, x) \in R_C^t$ implies

$$(y, x) \in R. \tag{7}$$

By definition of the transitive closure of a relation, $(x, y) \in \overline{R_C^t}$ implies that there exist $K \in \mathbb{N}$ and $x^0, \ldots, x^K \in X$ such that $x = x^0$, $(x^{k-1}, x^k) \in R_C^t$ for all $k \in \{1, \ldots, K\}$ and

$x^K = y$. By the induction hypothesis, $(x^{k-1}, x^k) \in R$ for all $k \in \{1, \ldots, K\}$. If $(x, y) \notin R$, (7) implies $(y, x) = (x^K, x^0) \in P(R)$. Because $(x^0, x^1) \in R$, we must have $K > 1$. But this contradicts the consistency of R. Therefore, $(x, y) \in R$.

To complete the proof of part (i), suppose $(x, y) \in R^*_C$. By definition, there exists $t \in \mathbb{N}_0$ such that $(x, y) \in R^t_C$ which, by the previous observation, implies $(x, y) \in R$.

The proof of part (ii) is analogous, given Lemma 1. ■

We now obtain

Theorem 3 *C satisfies* **G** *if and only if C satisfies consistent-closure coherence.*

Proof. To prove the only-if part of the theorem, suppose R is a consistent G-rationalization of C and let $S \in \Sigma$ and $x \in S$ be such that $(x, y) \in R^*_C$ for all $y \in S$. By Lemma 3, $(x, y) \in R$ for all $y \in S$. Thus, because R is a G-rationalization of C, $x \in C(S)$. Note that the consistency of R is not used in the above argument.

To establish the converse implication, we first prove that R^*_C is consistent (note that this is true by definition of R^*_C; consistent-closure coherence of C is not required). Suppose $K \in \mathbb{N} \setminus \{1\}$ and $x^0, \ldots, x^K \in X$ are such that $(x^{k-1}, x^k) \in R^*_C$ for all $k \in \{1, \ldots, K\}$. If $(x^K, x^0) \notin R^*_C$, we immediately obtain $(x^K, x^0) \notin P(R^*_C)$ and we are done. Now suppose that $(x^K, x^0) \in R^*_C$. By definition of R^*_C, it follows that, for all $k \in \{0, \ldots, K\}$, there exists $t^k \in \mathbb{N}_0$ such that $(x^{k-1}, x^k) \in R^{t^k}_C$ for all $k \in \{1, \ldots, K\}$ and $(x^K, x^0) \in R^{t^0}_C$. Let $t = \max\{t^k \mid k \in \{0, \ldots, K\}\}$. By (6), $(x^{k-1}, x^k) \in R^t_C$ for all $k \in \{1, \ldots, K\}$ and $(x^K, x^0) \in R^t_C$. This implies $(x^0, x^K) \in \overline{R^t_C}$ and, together with $(x^K, x^0) \in R^t_C$, the definition of R^{t+1}_C implies $(x^0, x^K) \in R^{t+1}_C$. From (6), it follows that $(x^0, x^K) \in R^*_C$ and thus $(x^K, x^0) \notin P(R^*_C)$, which establishes the consistency of R^*_C.

Now suppose C satisfies consistent-closure coherence. We complete the proof by showing that R^*_C is a G-rationalization of C. Let $S \in \Sigma$ and $x \in S$. Suppose $(x, y) \in R^*_C$ for all $y \in S$. Consistent-closure coherence implies $x \in C(S)$. Conversely, suppose $x \in C(S)$. By definition, this implies $(x, y) \in R_C$ for all $y \in S$ and, because $R_C = R^0_C \subseteq R^*_C$, we obtain $(x, y) \in R^*_C$ for all $y \in S$. ■

4 Binary Domains

In the case of binary domains, the presence of all two-element sets in Σ guarantees that every G-rationalization must be complete and, as a consequence, all rationality requirements involving greatest-element rationalizability and consistency become equivalent. In contrast, maximal-element rationalizability by a consistent and complete rationalization remains a stronger requirement than maximal-element rationalizability by a consistent and reflexive rationalization. These observations are summarized in the following theorem.

Theorem 4 *Suspose* Σ *is a binary domain. Then*

RCG, **CG**, **RCM**, **CM**, **RG**, **G**

$$\downarrow$$

RM, **M**

Proof. We divide the proof into the same three steps as in Theorem 1.

Step 1. We prove the equivalence of the axioms for each of the two boxes.

1.a. Using Theorem 1, the equivalence of the axioms in the top box follows from the observation that any consistent G-rationalization of C must be complete, given that Σ is binary.

1.b. This part is already proven in Theorem 1.

Step 2. Again, the strict implication indicated by the arrow in the theorem statement is straightforward.

Step 3. To prove that the reverse implication is not valid, Example 1 can be employed. ■

As shown in Theorem 4, there are only two different versions of rationalizability for binary domains. Consequently, we can restrict attention to the rationalizability axioms **G** and **M** in this case, keeping in mind that, by Theorem 4, all other rationalizability requirements involving consistent rationalizations are covered as well.

First, we show that **G** (and all other axioms that are equivalent to it according to Theorem 4) is characterized by the following weak congruence axiom (see Bossert, Sprumont and Suzumura, 2001).

Weak Congruence: For all $x, y, z \in X$, for all $S \in \Sigma$, if $(x, y) \in R_C$, $(y, z) \in R_C$, $x \in S$ and $z \in C(S)$, then $x \in C(S)$.

In contrast to congruence, weak congruence does not apply to chains of direct revealed preference of an arbitrary length but merely to chains involving three elements. For binary domains, weak congruence is necessary and sufficient for all forms of greatest-element rationalizability involving a consistent G-rationalization. It is an interesting observation that binary domains are sufficient to obtain results of that nature involving consistency, even though those domains do not necessarily contain all triples. This is in contrast to Sen's (1971) results which crucially depend on having all triples available in the domain. We obtain

Theorem 5 *Suppose* Σ *is a binary domain.* C *satisfies* **G** *if and only if* C *satisfies weak congruence.*

Proof. By Theorem 4, **G** is equivalent to **RCG** given that Σ is a binary domain. Moreover, as mentioned earlier, consistency is equivalent to transitivity in the presence of reflexivity and completeness. Theorem 3 in Bossert, Sprumont and Suzumura (2001) states that greatest-element rationalizability by a reflexive, complete and transitive relation is equivalent to weak congruence, provided that Σ is a binary domain. The result follows immediately as a consequence of this observation. ■

Finally, we establish that direct-revelation coherence and P-acyclicity of R_C together are necessary and sufficient for **M** (and **RM**) on a binary domain. This result is analogous to the characterization of greatest-element rationalizability by a P-acyclical, reflexive and complete rationalization on base domains (domains that contain all singletons in addition to all two-element sets) in Bossert, Sprumont and Suzumura (2001, Theorem 5).

Theorem 6 *Suppose* Σ *is a binary domain.* C *satisfies* **M** *if and only if* C *satisfies direct-revelation coherence and* R_C *is P-acyclical.*

Proof.

Step 1. We first show that **M** implies that R_C is P-acyclical (that direct-revelation coherence is implied follows from Lemma 2). Suppose R is a consistent M-rationalization of C. By way of contradiction, suppose R_C is not P-acyclical. Then there exist $K \in \mathbb{N} \setminus \{1\}$ and $x^0, \ldots, x^K \in X$ such that $(x^{k-1}, x^k) \in P(R_C)$ for all $k \in \{1, \ldots, K\}$ and $(x^K, x^0) \in P(R_C)$. Because Σ is a binary domain, $\{x^{k-1}, x^k\} \in \Sigma$ for all $k \in \{1, \ldots, K\}$ and $\{x^0, x^K\} \in \Sigma$. By definition of R_C, it follows that $x^k \notin C(\{x^{k-1}, x^k\})$ for all $k \in \{1, \ldots, K\}$ and $x^0 \notin C(\{x^0, x^K\})$. Because R is an M-rationalization of C, it follows that $(x^{k-1}, x^k) \in P(R)$ for all $k \in \{1, \ldots, K\}$ and $(x^K, x^0) \in P(R)$, contradicting the consistency of R.

Step 2. We show that direct-revelation coherence and the P-acyclicity of R_C together imply **M**. Define

$$R = R_C \setminus \{(x, y) \mid (x, y) \in I(R_C)\}.$$

By definition, $P(R) = R = P(R_C)$ and, consequently, R is consistent because R_C is P-acyclical.

It remains to be shown that R is an M-rationalization of C. Let $S \in \Sigma$ and $x \in S$. Suppose first that x is R-maximal in S, that is, $(y, x) \notin P(R)$ for all $y \in S$. If $S = \{x\}$, $x \in C(S)$ follows from the nonemptiness of $C(S)$. Now suppose $S \neq \{x\}$, and let $y \in S \setminus \{x\}$. Because Σ is a binary domain, $\{x, y\} \in \Sigma$. If $x \in C(\{x, y\})$, we obtain $(x, y) \in R_C$ by definition. If $x \notin C(\{x, y\})$, it follows that $(y, x) \in R_C$ and, because $(y, x) \notin P(R) = P(R_C)$ by assumption, we again obtain $(x, y) \in R_C$. By direct-revelation coherence, it follows that $x \in C(S)$.

Now suppose $x \in C(S)$. This implies $(x, y) \in R_C$ for all $y \in S$ and, therefore, $(y, x) \notin P(R_C) = P(R)$ for all $y \in S$. Therefore, x is R-maximal in S. ■

5 Concluding Remarks

The only notion of consistent rationalizability that is not characterized in this paper is maximal-element rationalizability by means of a consistent (and reflexive) rationalization on a general domain. The reason why it is difficult to obtain necessary and sufficient conditions in that case is the existential nature of the requirements for maximal-element rationalizability. It is immediately apparent that the revealed preference relation must be respected by any greatest-element rationalization, whereas this is not the case for maximal-element rationalizability (see Lemma 2). In order to exclude an element from a set of chosen alternatives according to maximal-element rationalizability, it merely is required that there exists (at least) *one* element in that set which is strictly preferred to the alternative to be excluded. The problem of identifying necessary and sufficient conditions for that kind of rationalizability is closely related to the problem of determining the *dimension* of a quasi-ordering; see, for example, Dushnik and Miller (1941). Because this is an area that is still quite unsettled, it is not too surprising that characterizations of maximal-element rationalizability on general domains are difficult to obtain. To the best of our knowledge, this is a feature that is shared by *all* notions of maximal-element rationalizability that are not equivalent to one of the notions of greatest-element rationalizability on general domains: we are not aware of any characterization results for maximal-element rationalizability on general domains unless the notion of maximal-element rationalizability employed happens to coincide with one of the notions of greatest-element rationalizability. Thus, there are important open questions to be addressed in future work in this area of research.

References

Arrow, K.J. (1951; second ed. 1963), *Social Choice and Individual Values,* John Wiley & Sons, New York.

Arrow, K.J. (1959), "Rational choice functions and orderings," *Economica* **26**, 121–127.

Bossert, W., Y. Sprumont and K. Suzumura (2001), "Rationalizability of choice functions on general domains without full transitivity," *Discussion Paper* **13-2001**, C.R.D.E., Université de Montréal.

Dushnik, B. and E.W. Miller (1941), "Partially ordered sets," *American Journal of Mathematics* **63**, 600–610.

Georgescu-Roegen, N. (1954), "Choice and revealed prference," *Southern Economic Journal* **21**, 119–130.

Georgescu-Roegen, N. (1966), *Analytical Economics: Issues and Problems,* Harvard University Press, Cambridge.

Hansson, B. (1968), "Choice structures and preference relations," *Synthese* **18**, 443–458.

Richter, M.K. (1966), "Revealed preference theory," *Econometrica* **34**, 635–645.

Richter, M.K. (1971), "Rational choice," in J. Chipman, L. Hurwicz, M. Richter, and H. Sonnenschein (eds.), *Preferences, Utility, and Demand,* Harcourt Brace Jovanovich, New York, pp. 29–58.

Samuelson, P.A. (1938), "A note on the pure theory of consumer's behaviour," *Economica* **5**, 61–71.

Samuelson, P.A. (1947), *Foundations of Economic Analysis,* Harvard University Press, Cambridge.

Samuelson, P.A. (1948), "Consumption theory in terms of revealed preference," *Economica* **15**, 243–253.

Samuelson, P.A. (1950), "The problem of integrability in utility theory," *Economica* **17**, 355–385.

SEN, A.K. (1971), "Choice functions and revealed preference," *Review of Economic Studies* **38**, 307–317.

SUZUMURA, K. (1976a), "Rational choice and revealed preference," *Review of Economic Studies* **43**, 149–158.

SUZUMURA, K. (1976b), "Remarks on the theory of collective choice," *Economica* **43**, 381–390.

SUZUMURA, K. (1977), "Houthakker's axiom in the theory of rational choice," *Journal of Economic Theory* **14**, 284–290.

SUZUMURA, K. (1983), *Rational Choice, Collective Decisions and Social Welfare,* Cambridge University Press, New York.