Maximal-Element Rationalizability[*]

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Abstract

We examine the maximal-element rationalizability of choice functions with arbitrary domains. While rationality formulated in terms of the choice of greatest elements according to a rationalizing relation has been analyzed relatively thoroughly in the earlier literature, this is not the case for maximal-element rationalizability, except when it coincides with greatest-element rationalizability because of properties imposed on the rationalizing relation. We develop necessary and sufficient conditions for maximal-element rationalizability by itself, and for maximal-element rationalizability in conjunction with additional properties of a rationalizing relation such as reflexivity, completeness, P-acyclicity, quasi-transitivity, consistency and transitivity. *Journal of Economic Literature* Classification No.: D11.

1   Introduction

The notion of rational choice as optimizing choice dates at least as far back as Robbins (1932; 1935, p. 93), who asserted that "there is a sense in which the word rationality can be used which renders it legitimate to argue that at least some rationality is assumed before human behaviour has an economic aspect—the sense, namely, in which it is equivalent to 'purposive'...." The elaborate edifice of revealed preference theory *à la* Samuelson (1938; 1947, Chapter V; 1948; 1950) and Houthakker (1950) was the first formal treatment of this notion of rational choice. The strong axiom of revealed preference due to Houthakker was meant to be a sufficient condition for the demand function of a competitive consumer to be derived by means of the optimization of an underlying preference ordering or utility function. This line of research has been further explored by Arrow (1959), Richter (1966; 1971), Hansson (1968), Sen (1971), Suzumura (1976; 1977; 1983, Chapter 2), Bossert, Sprumont and Suzumura (2001; 2002), and many others. Note, however, that the optimization of a single underlying preference ordering or utility function is not the only way of giving substance to the Robbinsian notion of 'purposive behaviour.' An alternative model of purposive behavior may require that there exist multiple preference orderings such that an alternative chosen from an option set is obtained by means of the maximization of the intersection of these underlying preference orderings. If these orderings may be construed as the individual preference orderings, the set of chosen options are nothing other than the set of Pareto-efficient options. Alternatively, the underlying preference orderings may be construed as potential preference orderings which a decision-maker may have in the future. In this case, the set of chosen options consists solely of those options which will never be rejected whichever potential preference ordering may materialize in the future. These examples will suffice to illustrate that the exploration of the Robbinsian notion of rational choice in terms of the maximal elements according to an underlying preference relation (which is not necessarily complete) is a worthwhile and important subject to be explored. See Schwartz (1976) and Sen (1997) for further motivation of exploring maximal-element rationalizability rather than greatest-element rationalizability. This paper is devoted to this issue. The analysis of necessary and sufficient conditions for maximal-element rationalizability by general relations and on arbitrary domains has, so far, not been explored thoroughly.

There are three identifiable domains of a choice function of historical importance. The first of these presupposes that the universal set is the *commodity space* (the non-negative orthant of some finite-dimensional Euclidean space) and requires that the domain of a

choice function consists of all *budget sets* (all non-degenerate subsets of the non-negative orthant whose northeastern boundary is a hyperplane with a positive normal). Samuelson (1938; 1947, Chapter V; 1948; 1950) and Houthakker (1950) developed a revealed preference theory for a competitive consumer under this domain restriction. Secondly, capitalizing on an acute observation by Georgescu-Roegen (1954, p. 125; 1966, p. 222), Arrow (1959) and Sen (1971) explored a new domain that includes all two-element sets and all three-element sets. The theory of rational choice thus developed served as one of the building blocks of non-binary social choice theory. See, for example, Sen (1977) and Suzumura (1983, Chapter 3). The third domain, which was introduced by Richter (1966; 1971) and Hansson (1968), imposes no extraneous restriction whatsoever on the class of feasible sets of options, thereby enabling the general theory of rational choice functions to be pursued. Along with Suzumura (1976; 1977; 1983, Chapter 2) and Bossert, Sprumont and Suzumura (2001; 2002), this paper attempts to explore the theory of rational choice functions on this general domain. As opposed to those earlier contributions, however, we focus on maximal-element rationalizability rather than greatest-element rationalizability. The only restrictions we impose throughout are that the domain be non-empty and that choices be *decisive*—that is, the set of chosen elements is always non-empty.

After formalizing alternative concepts of rationalizability in Section 2, we begin our analysis in Section 3 by examining the logical relationships which hold between the different notions of maximal-element rationalizability. These different versions of rationalizability are obtained if (combinations of) additional properties such as reflexivity, completeness, P-acyclicity, quasi-transitivity, consistency (in the sense of Suzumura, 1976; see Section 2 for a formal definition) and transitivity are imposed on rationalizing relations. For each notion of maximal-element rationalizability, we provide a set of necessary and sufficient conditions. In particular, Section 4 presents complete characterizations of those notions of maximal-element rationalizability that are weaker than full rationalizability (that is, rationalizability by an ordering, in which case maximal elements and greatest elements coincide). Because of the different nature of full rationalizability (and the different nature of the characterizing conditions involved), this form of rationalizability is analyzed in a section of its own. In Section 5, we provide a new characterization that is formulated in a framework analogous to that used in the previous section. This result provides an important link between our approach and the earlier analysis of full rationalizability (in particular, the axiomatization due to Richter, 1966), and it serves to illustrate how our contribution fits into the existing literature. Section 6 shows how the result of Section 5 can be simplified if the choice function is single-valued. Section 7 concludes with remarks

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on some further problems to be explored.

2   Alternative Concepts of Rationalizability

The set of positive integers is denoted by $\mathbb{N}$. Let X be a universal non-empty set of alternatives. $\mathcal{X}$ is the power set of X excluding the empty set. A *choice function* is a mapping $C: \Sigma \to \mathcal{X}$ such that $C(S) \subseteq S$ for all $S \in \Sigma$, where $\Sigma \subseteq \mathcal{X}$ with $\Sigma \neq \emptyset$ is the domain of C. Note that C maps Σ into the set of all *non-empty* subsets of X. Thus, according to Richter's (1971) terminology, the choice function C is *decisive*. No restrictions other than non-emptiness are imposed on the universal set X, or on the domain of the choice function Σ. The alternatives $x^0, \ldots, x^K \in X$ with $K \in \mathbb{N}$ are said to form a *revealed preference chain of order K* if there exist $S^1, \ldots, S^K \in \Sigma$ such that $x^{k-1} \in C(S^k)$ and $x^k \in S^k$ for all $k \in \{1, \ldots, K\}$.

Let $R \subseteq X \times X$ be a (binary) relation on X. The *asymmetric factor $P(R)$* of R is defined by $(x, y) \in P(R)$ if and only if $(x, y) \in R$ and $(y, x) \notin R$ for all $x, y \in X$. The *symmetric factor $I(R)$* of R is defined by $(x, y) \in I(R)$ if and only if $(x, y) \in R$ and $(y, x) \in R$ for all $x, y \in X$. The *non-comparable factor $N(R)$* of R is defined by $(x, y) \in N(R)$ if and only if $(x, y) \notin R$ and $(y, x) \notin R$ for all $x, y \in X$. The *diagonal relation* on X is given by $R_d = \{(x, x) \mid x \in X\}$.

A relation $R \subseteq X \times X$ is (i) *reflexive* if, for all $x \in X$, $(x, x) \in R$; (ii) *complete* if, for all $x, y \in X$ such that $x \neq y$, $(x, y) \in R$ or $(y, x) \in R$; (iii) *P-acyclical* if, for all $K \in \mathbb{N} \setminus \{1\}$ and for all $x^0, \ldots, x^K \in X$, $(x^{k-1}, x^k) \in P(R)$ for all $k \in \{1, \ldots, K\}$ implies $(x^K, x^0) \notin P(R)$; (iv) *quasi-transitive* if, for all $x, y, z \in X$, $[(x, y) \in P(R)$ and $(y, z) \in P(R)]$ implies $(x, z) \in P(R)$; (v) *consistent* if, for all $K \in \mathbb{N} \setminus \{1\}$ and for all $x^0, \ldots, x^K \in X$, $(x^{k-1}, x^k) \in R$ for all $k \in \{1, \ldots, K\}$ implies $(x^K, x^0) \notin P(R)$; (vi) *transitive* if, for all $x, y, z \in X$, $[(x, y) \in R$ and $(y, z) \in R]$ implies $(x, z) \in R$.

Note that reflexivity is equivalent to the set inclusion $R_d \subseteq R$. Furthermore, a transitive relation is consistent, and a consistent relation is P-acyclical. Transitivity implies quasi-transitivity which, in turn, implies P-acyclicity. The reverse implications are not true in general. However, the discrepancy between transitivity and consistency disappears if the relation is reflexive and complete. See Suzumura (1983, p. 244). Consistency and quasi-transitivity are independent. A reflexive, complete and transitive relation is called an *ordering*.

The *transitive closure* of a relation $R \subseteq X \times X$ is denoted by $\overline{R}$, that is, for all $x, y \in X$, $(x, y) \in \overline{R}$ if there exist $K \in \mathbb{N}$ and $x^0, \ldots, x^K \in X$ such that $x = x^0$, $(x^{k-1}, x^k) \in R$ for

all $k \in \{1, \dots, K\}$ and $x^K = y$. Clearly, $\overline{R}$ is transitive and, because we can set $K = 1$, it follows that $R \subseteq \overline{R}$.

For a set $S \in \Sigma$ and a relation $R \subseteq X \times X$, the set of R-maximal elements in S is

$$M(S, R) = \{x \in S \mid (y, x) \notin P(R) \text{ for all } y \in S\}.$$

A choice function C is *maximal-element rationalizable, M-rationalizable* for short, if there exists a relation R on X, to be called an *M-rationalization* of C, such that $C(S) = M(S, R)$ for all $S \in \Sigma$.

Even though this paper is primarily concerned with maximal-element rationalizability, a by-product of our analysis pertains to greatest-element rationalizability. Thus, we also define the set of R-greatest elements in S as

$$G(S, R) = \{x \in S \mid (x, y) \in R \text{ for all } y \in S\},$$

and a choice function C is *greatest-element rationalizable, G-rationalizable* for short, if there exists a relation R on X, to be called a *G-rationalization* of C, such that $C(S) = G(S, R)$ for all $S \in \Sigma$. Note that $G(S, R) \subseteq M(S, R)$ holds for all $S \in \Sigma$ and for any relation R, where the set inclusion must be satisfied with an equality if R is reflexive and complete.
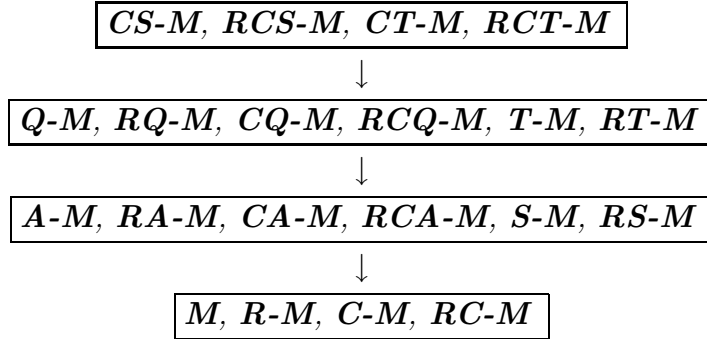
Depending on the additional properties that we might want to impose on a rationalization (if any), different notions of rationalizability can be defined. For simplicity of presentation, we use the following terminology. **M** stands for maximal-element rationalizability by an arbitrary M-rationalization, and **R-M** (respectively **C-M**; **RC-M**) is maximal-element rationalizability by means of a reflexive (respectively complete; reflexive and complete) M-rationalization. **A-M** (respectively **RA-M**; **CA-M**; **RCA-M**) is maximal-element rationalizability by a P-acyclical (respectively reflexive and P-acyclical; complete and P-acyclical; reflexive, complete and P-acyclical) M-rationalization, and **Q-M** (respectively **RQ-M**; **CQ-M**; **RCQ-M**) is maximal-element rationalizability by a quasi-transitive (respectively reflexive and quasi-transitive; complete and quasi-transitive; reflexive, complete and quasi-transitive) M-rationalization. Furthermore, **S-M** (respectively **RS-M**; **CS-M**; **RCS-M**) represents maximal-element rationalizability by a consistent (respectively reflexive and consistent; complete and consistent; reflexive, complete and consistent) M-rationalization. Analogously, **T-M** (respectively **RT-M**; **CT-M**; **RCT-M**) denotes maximal-element rationalizability by a transitive (respectively reflexive and transitive; complete and transitive; reflexive, complete and transitive) M-rationalization. Finally, **RC-G** (respectively **RCT-G**) is greatest-element rationaliz-

ability by a reflexive and complete (respectively reflexive, complete and transitive) G-rationalization. In particular, we refer to **RCT-G** (and all notions that are equivalent to it) as *full rationalizability.* All weaker rationalizability requirements are collected under the term *weak rationalizability.*

3   Logical Relationships

We begin our analysis by providing a full description of the logical relationships which hold between the different notions of maximal-element rationalizability that can be defined, depending on which additional properties are imposed on an M-rationalization. For convenience, a diagrammatic representation is employed: all axioms that are depicted within the same box are equivalent, and an arrow pointing from one box b to another box b' indicates that the axioms in b imply those in b', and the converse implication is not true without further assumptions regarding the domain of C.

Theorem 1 *Suppose Σ is an arbitrary non-empty domain. Then*

$$\boxed{\textbf{CS-M},\ \textbf{RCS-M},\ \textbf{CT-M},\ \textbf{RCT-M}}$$
$$\downarrow$$
$$\boxed{\textbf{Q-M},\ \textbf{RQ-M},\ \textbf{CQ-M},\ \textbf{RCQ-M},\ \textbf{T-M},\ \textbf{RT-M}}$$
$$\downarrow$$
$$\boxed{\textbf{A-M},\ \textbf{RA-M},\ \textbf{CA-M},\ \textbf{RCA-M},\ \textbf{S-M},\ \textbf{RS-M}}$$
$$\downarrow$$
$$\boxed{\textbf{M},\ \textbf{R-M},\ \textbf{C-M},\ \textbf{RC-M}}$$

Proof. The proof is organized as follows. In Step 1, we prove the equivalence of all axioms that appear in the same box. In Step 2, we show that all implications depicted in the theorem statement are valid. In Step 3, we provide examples demonstrating that no further implications are true without additional assumptions.

Step 1 For each of the four boxes, we show that all axioms listed in the box are equivalent.

1.a To establish the equivalence of the axioms in the top box, it is sufficient to show that **CS-M** implies **RCT-M**. Suppose R is a complete and consistent M-rationalization of C. Let $R' = R \cup R_d$. Clearly, R' is reflexive. Furthermore, R' is complete because R is. Because consistency is equivalent to transitivity for a reflexive and complete relation, R' is

transitive. That R' is an M-rationalization of C follows immediately from the assumption that R is, given the definition of maximal-element rationalizability.

1.b The equivalences in the second box are established in Bossert, Sprumont and Suzumura (2001).

1.c Now consider the third box. Clearly, it is sufficient to prove that **RS-M** implies **RCA-M** and that **A-M** implies **RS-M**.

To prove the first implication, suppose R is a reflexive and consistent rationalization of C. As is straightforward to verify, the relation $R' = R \cup N(R)$ is reflexive and complete. Furthermore, because $P(R') = P(R)$, it follows immediately that R' is an M-rationalization of C.

Now suppose R is a P-acyclical M-rationalization of C. Defining $R' = (R \setminus I(R)) \cup R_d$, it follows immediately that R' is a reflexive and consistent M-rationalization of C.

1.d The equivalence of the properties in the fourth box is established by showing that **M** implies **RC-M**. Suppose R is an M-rationalization of C. Let $R' = R \cup N(R)$. As in Step 1.c, in view of $P(R') = P(R)$, it follows immediately that R' is a reflexive and complete M-rationalization of C.

Step 2 The strict implications depicted by the arrows in the theorem statement are straightforward to verify.

Step 3 Given the equivalences established in Step 1, the examples defined in Steps 3.a to 3.c suffice to complete the proof of the theorem.

3.a T-M does not imply **CT-M**.

Example 1 *Let $X = \{x, y, z\}$ and $\Sigma = \{\{x, y\}, \{x, z\}, \{y, z\}\}$. Define the choice function C by letting $C(\{x, y\}) = \{x, y\}$, $C(\{x, z\}) = \{z\}$ and $C(\{y, z\}) = \{y, z\}$. This choice function is M-rationalizable by the transitive M-rationalization*

$$R = \{(z, x)\}.$$

Suppose C is M-rationalizable by a complete and transitive M-rationalization R'. By definition of maximal-element rationalizability, we must have $(z, x) \in P(R')$ because $x \notin C(\{x, z\})$. Completeness of R' implies, together with the definition of maximal-element rationalizability, that we must have $(x, y) \in I(R')$ and $(y, z) \in I(R')$. By transitivity of R', it follows that $(x, z) \in I(R')$, a contradiction.

3.b S-M does not imply **T-M**.

Example 2 *Let $X = \{x, y, z\}$ and $\Sigma = \{\{x, y\}, \{x, z\}, \{y, z\}\}$, and define $C(\{x, y\}) = \{x\}$, $C(\{x, z\}) = \{x, z\}$ and $C(\{y, z\}) = \{y\}$. This choice function is M-rationalizable*

by the consistent M-rationalization

$$R = \{(x, y), (y, z)\}.$$

Suppose R' is a transitive M-rationalization of C. Because $z \in C(\{x, z\})$, the definition of maximal-element rationalizability implies $(x, z) \notin P(R')$. Again using the definition of maximal-element rationalizability, we must have $(x, y) \in P(R')$ because $y \notin C(\{x, y\})$ and $(y, z) \in P(R')$ because $z \notin C(\{y, z\})$. The transitivity of R' implies $(x, z) \in P(R')$, a contradiction.

3.c M does not imply **S-M**.

Example 3 *Let $X = \{x, y, z\}$ and $\Sigma = \{\{x, y\}, \{x, z\}, \{y, z\}\}$, and define $C(\{x, y\}) = \{x\}$, $C(\{x, z\}) = \{z\}$ and $C(\{y, z\}) = \{y\}$. This choice function is M-rationalizable by the M-rationalization*

$$R = \{(x, y), (y, z), (z, x)\}.$$

Suppose R' is an M-rationalization of C. Because $y \notin C(\{x, y\})$, the definition of maximal-element rationalizability implies $(x, y) \in P(R')$. Analogously, $x \notin C(\{x, z\})$ implies $(z, x) \in P(R')$, and $z \notin C(\{y, z\})$ implies $(y, z) \in P(R')$. This implies that R' is not consistent.

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4   Weak Forms of M-Rationalizability

We now provide characterizations of the three weakest notions of M-rationalizability identified in Theorem 1. In addition, we prove a result that employs the remaining combination of the axioms considered in this section.

As as auxiliary step in formulating various sets of necessary and sufficient conditions, we introduce some further definitions. Let

$$\mathcal{A}_C = \{(S, y) \mid S \in \Sigma \text{ and } y \in S \setminus C(S)\}$$

and

$$\mathcal{F}_C = \{f \colon \mathcal{A}_C \to X \mid f(S, y) \in S \text{ for all } (S, y) \in \mathcal{A}_C\}.$$

The set $\mathcal{A}_C$ contains all pairs consisting of a feasible set and an element that belongs to the set but is not chosen by C. The only case where $\mathcal{A}_C$ is empty occurs if $C(S) = S$

for all $S \in \Sigma$, that is, all feasible elements are chosen in each and every choice situation. The functions in $\mathcal{F}_C$ also have an intuitive interpretation. They assign a feasible element to each pair of a feasible set and an alternative that is not chosen therefrom. Within our framework of maximal-element rationalizability, the intended interpretation is that $f(S, y)$ is an alternative in S that can be used to prevent y from being chosen. Clearly, the existence of such an alternative for each (S, y) in $\mathcal{A}_C$ is a necessary condition for maximal-element rationalizability.

The following properties of a function $f \in \mathcal{F}_C$ will be of importance in formulating our conditions.

A For all $(S, y) \in \mathcal{A}_C$, for all $T \in \Sigma$ and for all $x \in X$,

$$[f(S, y) = x \text{ and } x \in T] \Rightarrow y \notin C(T).$$

$\overline{\mathbf{A}}$ For all $K \in \mathbb{N}$, for all $(S^1, x^1), \ldots, (S^K, x^K) \in \mathcal{A}_C$, for all $S^0 \in \Sigma$ and for all $x^0 \in S^0$,

$$f(S^k, x^k) = x^{k-1} \text{ for all } k \in \{1, \ldots, K\} \Rightarrow x^K \notin C(S^0).$$

B For all $(S, y), (T, x) \in \mathcal{A}_C$,

$$f(S, y) = x \Rightarrow f(T, x) \neq y.$$

$\overline{\mathbf{B}}$ For all $K \in \mathbb{N}$, for all $(S^0, x^0), \ldots, (S^K, x^K) \in \mathcal{A}_C$,

$$f(S^k, x^k) = x^{k-1} \text{ for all } k \in \{1, \ldots, K\} \Rightarrow f(S^0, x^0) \neq x^K.$$

The properties **A** and $\overline{\mathbf{A}}$ impose restrictions on the relationship between C and f, whereas **B** and $\overline{\mathbf{B}}$ are concerned with avoiding contradictory behavior of the function f itself. Clearly, $\overline{\mathbf{A}}$ implies **A** and $\overline{\mathbf{B}}$ implies **B** by definition.

These properties enable us to introduce several axioms which completely characterize various concepts of M-rationalizability. The $\overline{\mathbf{A}}\mathbf{B}$-axiom below is introduced for completeness of the analysis to be carried out in this section. Although it does not characterize any of the rationalizability requirements introduced so far in this paper, it is worthwhile to examine its consequences by characterizing all choice functions that satisfy the axiom.

For simplicity of exposition, we only mention one rationalizability property out of each set of equivalent properties; clearly, additional equivalence results are obtained by applying Theorem 1.

To begin with, the following axiom is necessary and sufficient for maximal-element rationalizability on an arbitrary domain.

AB-axiom: If $\mathcal{A}_C \neq \emptyset$, then there exists $f \in \mathcal{F}_C$ satisfying **A** and **B**.

We obtain the following characterization result.

Theorem 2 *C satisfies M if and only if C satisfies the **AB**-axiom.*

Proof. Step 1 We first prove that **M** implies the **AB**-axiom. Let R be an M-rationalization of C. If $\mathcal{A}_C = \emptyset$, the **AB**-axiom is obviously satisfied.

Now suppose $\mathcal{A}_C \neq \emptyset$. To define a function $f \colon \mathcal{A}_C \to X$ with the desired properties, consider any $(S, y) \in \mathcal{A}_C$. By definition, $S \in \Sigma$ and $y \in S \setminus C(S)$. The assumption that R maximal-element rationalizes C implies the existence of $x \in S$ such that $(x, y) \in P(R)$. Define $f(S, y) = x$. Clearly, $f(S, y) \in S$ for all $(S, y) \in \mathcal{A}_C$ by definition and, thus, $f \in \mathcal{F}_C$. We show that the function f satisfies **A** and **B**.

To establish **A**, suppose $(S, y) \in \mathcal{A}_C$, $T \in \Sigma$ and $x \in X$ are such that $f(S, y) = x$ and $x \in T$. By the above definition of f, we obtain $(x, y) \in P(R)$. Because R is an M-rationalization of C, it follows that $y \notin C(T)$.

To establish **B**, let $(S, y), (T, x) \in \mathcal{A}_C$ and suppose $f(S, y) = x$. The definition of f again implies $(x, y) \in P(R)$. If $f(T, x) = y$, the same reasoning yields $(y, x) \in P(R)$, a contradiction. Thus, $f(T, x) \neq y$, establishing **B**.

Step 2 The proof is completed by establishing that the **AB**-axiom implies **M**. Suppose C satisfies the **AB**-axiom. We construct an M-rationalization R of C. If $\mathcal{A}_C = \emptyset$, any relation R such that $P(R) = \emptyset$ is an M-rationalization of C.

If $\mathcal{A}_C \neq \emptyset$, the **AB**-axiom implies the existence of a function $f \in \mathcal{F}_C$ satisfying **A** and **B**.

Define $R = \{(f(S, y), y) \mid (S, y) \in \mathcal{A}_C\}$. It remains to be shown that R is an M-rationalization of C. Let $S \in \Sigma$ and $x \in S$.

Suppose $x \in C(S)$. If there exists $y \in S$ such that $(y, x) \in P(R)$, it follows from the definition of R that there exists $T \in \Sigma$ such that $(T, x) \in \mathcal{A}_C$ and $f(T, x) = y$. But this contradicts **A** and, therefore, x is R-maximal in S. Hence, $C(S) \subseteq M(S, R)$.

Now suppose $x \notin C(S)$. Let $y = f(S, x)$. By definition of R, we obtain $(y, x) \in R$. By way of contradiction, suppose we also have $(x, y) \in R$. Then there exists $T \in \Sigma$ such that $(T, y) \in \mathcal{A}_C$ and $f(T, y) = x$. But this contradicts **B**. Therefore, $(x, y) \notin R$ and thus $(y, x) \in P(R)$. Hence, x is not R-maximal in S, and we obtain $M(S, R) \subseteq C(S)$. ■

The following corollary is an immediate consequence of Theorem 2.

Corollary 1 *C satisfies RC-G if and only if C satisfies the **AB**-axiom.*

To establish this corollary to Theorem 2, we have only to recollect that $M(S, R) = G(S, R)$ holds for all $S \in \Sigma$ if R is reflexive and complete. Simple though this corollary is, it solves a perennial problem left open by Richter (1971, p. 36).[1] Thus, our study of M-rationalizability casts some light on the theory of G-rationalizability as a by-product.

Back, then, to the theory of M-rationalizability. The following condition characterizes M-rationalizability by a P-acyclical M-rationalization.

$\mathbf{A}\overline{\mathbf{B}}$: If $\mathcal{A}_C \neq \emptyset$, then there exists $f \in \mathcal{F}_C$ satisfying $\mathbf{A}$ and $\overline{\mathbf{B}}$.

We now obtain

Theorem 3 *C satisfies $\mathbf{A}\text{-}\mathbf{M}$ if and only if C satisfies the $\mathbf{A}\overline{\mathbf{B}}$-axiom.*

Proof. Step 1 We first prove that $\mathbf{A}\text{-}\mathbf{M}$ implies the $\mathbf{A}\overline{\mathbf{B}}$-axiom. Let R be a P-acyclical M-rationalization of C. If $\mathcal{A}_C = \emptyset$, the proof of Step 1 is complete.

Now suppose $\mathcal{A}_C \neq \emptyset$. To define a function $f \colon \mathcal{A}_C \to X$ with the desired properties, consider any $(S, y) \in \mathcal{A}_C$. By definition, $S \in \Sigma$ and $y \in S \setminus C(S)$. The assumption that R maximal-element rationalizes C implies the existence of $x \in S$ such that $(x, y) \in P(R)$. Define $f(S, y) = x$. Again, it is clear that $f(S, y) \in S$ for all $(S, y) \in \mathcal{A}_C$. That $\mathbf{A}$ is satisfied by the function f follows as in the proof of Theorem 2.

To establish $\overline{\mathbf{B}}$, suppose $K \in \mathbb{N}$ and $(S^0, x^0), \ldots, (S^K, x^K) \in \mathcal{A}_C$ are such that $f(S^k, x^k) = x^{k-1}$ for all $k \in \{1, \ldots, K\}$. By definition of f, we obtain $(x^{k-1}, x^k) \in P(R)$ for all $k \in \{1, \ldots, K\}$. If $f(S^0, x^0) = x^K$, it follows that $(x^K, x^0) \in P(R)$. If $K = 1$, this contradicts the hypothesis $(x^0, x^K) \in P(R)$ and if $K > 1$, we obtain a contradiction to the P-acyclicity of R. Therefore, $f(S^0, x^0) \neq x^K$.

Step 2 The proof is completed by establishing that the $\mathbf{A}\overline{\mathbf{B}}$-axiom implies $\mathbf{A}\text{-}\mathbf{M}$. Suppose C satisfies the $\mathbf{A}\overline{\mathbf{B}}$-axiom. If $\mathcal{A}_C = \emptyset$, we are done.

If $\mathcal{A}_C \neq \emptyset$, the $\mathbf{A}\overline{\mathbf{B}}$-axiom implies the existence of a function $f \in \mathcal{F}_C$ satisfying $\mathbf{A}$ and $\overline{\mathbf{B}}$.

Define $R = \{(f(S, y), y) \mid (S, y) \in \mathcal{A}_C\}$. That R is an M-rationalization of C follows as in the proof of Theorem 2. To show that R is P-acyclical, suppose $K \in \mathbb{N} \setminus \{1\}$ and $x^0, \ldots, x^K \in X$ are such that $(x^{k-1}, x^k) \in P(R)$ for all $k \in \{1, \ldots, K\}$. By definition of R, this implies that there exist $S^1, \ldots, S^K \in \Sigma$ such that $(S^k, x^k) \in \mathcal{A}_C$ and $x^{k-1} = f(S^k, x^k)$

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[1]Richter's (1971, p. 36) Theorem 7 shows that a G-rational choice function with a complete G-rationalization also has a complete and reflexive G-rationalization. However, he also observed that "[i]t would be nice to have a behavioral characterization of [G-rational choice function with complete G-rationalizations], but this remains an open problem." Our Corollary 1 is a solution of this open problem.

for all $k \in \{1, \ldots, K\}$. If $(x^K, x^0) \in P(R)$, there exists $S^0 \in \Sigma$ such that $(S^0, x^0) \in \mathcal{A}_C$ and $x^K = f(S^0, x^0)$. But this contradicts $\overline{\mathbf{B}}$. ∎

Likewise, transitive M-rationalizability is characterized by the following axiom.

$\overline{\mathbf{AB}}$-axiom: If $\mathcal{A}_C \neq \emptyset$, then there exists $f \in \mathcal{F}_C$ satisfying $\overline{\mathbf{A}}$ and $\overline{\mathbf{B}}$.

The corresponding characterization result is stated in the following theorem.

Theorem 4 C *satisfies* $\boldsymbol{Q\text{-}M}$ *if and only if* C *satisfies the* $\overline{\mathbf{AB}}$*-axiom.*

Proof. Step 1 We first prove that $\mathbf{Q\text{-}M}$ implies the $\overline{\mathbf{AB}}$-axiom. Let R be a transitive M-rationalization of C. If $\mathcal{A}_C = \emptyset$, the proof of Step 1 is complete.

Now suppose $\mathcal{A}_C \neq \emptyset$. To define a function $f \colon \mathcal{A}_C \to X$ with the desired properties, consider any $(S, y) \in \mathcal{A}_C$. By definition, $S \in \Sigma$ and $y \in S \setminus C(S)$. The assumption that R maximal-element rationalizes C implies the existence of $x \in S$ such that $(x, y) \in P(R)$. Define $f(S, y) = x$. Again, it is clear that $f(S, y) \in S$ for all $(S, y) \in \mathcal{A}_C$.

To show that f satisfies $\overline{\mathbf{A}}$, suppose $K \in \mathbb{N}$, $(S^1, x^1), \ldots, (S^K, x^K) \in \mathcal{A}_C$, $S^0 \in \Sigma$ and $x^0 \in S^0$ are such that $f(S^k, x^k) = x^{k-1}$ for all $k \in \{1, \ldots, K\}$. By definition, $(x^{k-1}, x^k) \in P(R)$ for all $k \in \{1, \ldots, K\}$. By quasi-transitivity of R, $(x^0, x^K) \in P(R)$. Because R is an M-rationalization of C, it follows that $x^K \notin C(S^0)$.

To prove $\overline{\mathbf{B}}$, suppose $K \in \mathbb{N}$ and $(S^0, x^0), \ldots, (S^K, x^K) \in \mathcal{A}_C$ are such that $f(S^k, x^k) = x^{k-1}$ for all $k \in \{1, \ldots, K\}$. By definition of f, we obtain $(x^{k-1}, x^k) \in P(R)$ for all $k \in \{1, \ldots, K\}$. Because R is quasi-transitive, we obtain $(x^0, x^K) \in P(R)$ and hence $(x^K, x^0) \notin P(R)$. By definition of f, this implies $f(S^0, x^0) \neq x^K$.

Step 2 The proof is completed by establishing that the $\overline{\mathbf{AB}}$-axiom implies $\mathbf{Q\text{-}M}$. Suppose C satisfies the $\overline{\mathbf{AB}}$-axiom. If $\mathcal{A}_C = \emptyset$, we are done.

If $\mathcal{A}_C \neq \emptyset$, the $\overline{\mathbf{AB}}$-axiom implies the existence of a function $f \in \mathcal{F}_C$ satisfying $\overline{\mathbf{A}}$ and $\overline{\mathbf{B}}$.

Define $R = \{(f(S, y), y) \mid (S, y) \in \mathcal{A}_C\}$ and consider the transitive closure $\overline{R}$ of R. We show that $\overline{R}$ is an M-rationalization of C. Let $S \in \Sigma$ and $x \in S$.

Suppose $x \in C(S)$. If there exists $y \in S$ such that $(y, x) \in P(\overline{R})$, it follows that there exist $K \in \mathbb{N}$ and $(S^1, x^1), \ldots, (S^K, x^K) \in \mathcal{A}_C$ such that, with $x^0 = y$ and $x^K = x$, $x^{k-1} = f(S^k, x^k)$ for all $k \in \{1, \ldots, K\}$. Letting $S^0 = S$, we obtain a contradiction to $\overline{\mathbf{A}}$ and, thus, x is $\overline{R}$-maximal in S. Hence, $C(S) \subseteq M(S, \overline{R})$.

Finally, suppose $x \notin C(S)$. Let $y = f(S, x)$. By definition of R and the transitive closure of a relation, this implies $(y, x) \in R \subseteq \overline{R}$. Suppose $(x, y) \in \overline{R}$. Then there

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exist $K \in \mathbb{N}$ and $(S^0, x^0), \ldots, (S^K, x^K) \in \mathcal{A}_C$ such that $(S^0, x^0) = (S, x)$, $x^K = y$ and $f(S^k, x^k) = x^{k-1}$ for all $k \in \{1, \ldots, K\}$. By $\overline{\mathbf{B}}$, $f(S^0, x^0) = f(S, x) \neq x^K = y$, a contradiction. Therefore, $(x, y) \notin \overline{R}$ and hence $(y, x) \in P(\overline{R})$. Thus, x is not $\overline{R}$-maximal in S and, therefore, $M(S, \overline{R}) \subseteq C(S)$. ∎

We conclude this section by characterizing all choice functions such that there exists a function $f \in \mathcal{F}_C$ satisfying $\overline{\mathbf{A}}$ and $\mathbf{B}$ whenever $\mathcal{A}_C$ is non-empty. We say that a choice function satisfies $\mathbf{U}$-$\mathbf{M}$ if it possesses an M-rationalization R such that, for all $K \in \mathbb{N}$, for all $x^1, \ldots, x^K \in X$, for all $S^0 \in \Sigma$ and for all $x^0 \in S^0$,

$$(x^{k-1}, x^k) \in P(R) \text{ for all } k \in \{1, \ldots, K\} \Rightarrow x^K \notin C(S^0).$$

Furthermore, we define the following $\overline{\mathbf{A}}\mathbf{B}$-axiom.

$\overline{\mathbf{A}}\mathbf{B}$-axiom: If $\mathcal{A}_C \neq \emptyset$, then there exists $f \in \mathcal{F}_C$ satisfying $\overline{\mathbf{A}}$ and $\mathbf{B}$.

We conclude this section with a characterization of $\mathbf{U}$-$\mathbf{M}$.

Theorem 5 *C satisfies $\mathbf{U}$-$\mathbf{M}$ if and only if C satisfies the $\overline{\mathbf{A}}\mathbf{B}$-axiom.*

Proof. Step 1 We first prove that $\mathbf{U}$-$\mathbf{M}$ implies the $\overline{\mathbf{A}}\mathbf{B}$-axiom. Let R be an M-rationalization of C satisfying the property in the definition of $\mathbf{U}$-$\mathbf{M}$. If $\mathcal{A}_C = \emptyset$, the proof of Step 1 is complete.

Now suppose $\mathcal{A}_C \neq \emptyset$. To define a function $f: \mathcal{A}_C \to X$ with the desired properties, consider any $(S, y) \in \mathcal{A}_C$. By definition, $S \in \Sigma$ and $y \in S \setminus C(S)$. The assumption that R maximal-element rationalizes C implies the existence of $x \in S$ such that $(x, y) \in P(R)$. Define $f(S, y) = x$. It follows that $f(S, y) \in S$ for all $(S, y) \in \mathcal{A}_C$.

To prove that f satisfies $\overline{\mathbf{A}}$, suppose $K \in \mathbb{N}$, $(S^1, x^1), \ldots, (S^K, x^K) \in \mathcal{A}_C$, $S^0 \in \Sigma$ and $x^0 \in S^0$ are such that $f(S^k, x^k) = x^{k-1}$ for all $k \in \{1, \ldots, K\}$. By definition, $(x^{k-1}, x^k) \in P(R)$ for all $k \in \{1, \ldots, K\}$. By $\mathbf{U}$-$\mathbf{M}$, it follows that $x^K \notin C(S^0)$. That $\mathbf{B}$ is satisfied follows as in the proof of Theorem 2.

Step 2 The proof is completed by establishing that the $\overline{\mathbf{A}}\mathbf{B}$-axiom implies $\mathbf{U}$-$\mathbf{M}$. Suppose C satisfies the $\overline{\mathbf{A}}\mathbf{B}$-axiom. If $\mathcal{A}_C = \emptyset$, we are done.

If $\mathcal{A}_C \neq \emptyset$, the $\overline{\mathbf{A}}\mathbf{B}$-axiom implies the existence of a function $f \in \mathcal{F}_C$ satisfying $\overline{\mathbf{A}}$ and $\mathbf{B}$.

Define $R = \{(f(S, y), y) \mid (S, y) \in \mathcal{A}_C\}$. That R is an M-rationalization of C follows as in the proof of Theorem 2. To complete the proof, we have to establish the property in the definition of $\mathbf{U}$-$\mathbf{M}$. Suppose $K \in \mathbb{N}$, $x^1, \ldots, x^K \in X$, $S^0 \in \Sigma$ and $x^0 \in S^0$ are such

that $(x^{k-1}, x^k) \in P(R)$ for all $k \in \{1, \ldots, K\}$. By definition, there exist $S^1, \ldots, S^K \in \Sigma$ such that $(S^k, x^k) \in \mathcal{A}_C$ and $x^{k-1} = f(S^k, x^k)$ for all $k \in \{1, \ldots, K\}$. By $\overline{\mathbf{A}}$, $x^K \notin C(S^0)$. ∎

5 Full Rationalizability

In general, the crucial feature of maximal-element rationalizability is to establish instances of strict preference which prevent non-chosen alternatives to be maximal elements. As is easy to verify, maximal-element rationalizability coincides with greatest-element rationalizability if a rationalization is required to be reflexive and complete. If transitivity is imposed in addition (and, thus, full rationalizability is considered), instances of indifference are important in addition to strict preferences; see Houthakker (1950) and Suzumura (1977) for discussions. In our present context, it is therefore clear that the functions in $\mathcal{F}_C$—the interpretation of which is concerned exclusively with strict preferences—are no longer sufficient to provide an adequate framework for the formulation of necessary and sufficient conditions for full rationalizability. However, we may preserve the general nature of our approach and work with the following modifications of the analytical framework in the previous section. Let

$$\mathcal{B}_C = \{(S, y) \mid S \in \Sigma \text{ and } y \in S\}$$

and

$$\mathcal{G}_C = \{g \colon \mathcal{B}_C \to X \mid g(S, y) \in C(S) \text{ for all } (S, y) \in \mathcal{B}_C\}.$$

The set $\mathcal{B}_C$ contains all pairs consisting of a feasible set S and an alternative y that is in S—y may or may not be in $C(S)$. It follows that $\mathcal{B}_C$ is non-empty because the domain of C is non-empty and contains only non-empty subsets of X. Each of the functions in $\mathcal{G}_C$ assigns a chosen alternative to each pair of a feasible set and a feasible alternative. The intended interpretation is that $g(S, y) \in S$ represents a selection from $C(S)$ that cannot be prevented from being chosen by the presence of y.

Due to the more demanding nature of full rationalizability, the two separate types of properties used in the previous section coincide and can be expressed in terms of a single condition. We define the following restriction on g.

$\overline{\mathbf{D}}$ For all $K \in \mathbb{N}$ and for all $(S^0, x^0), \ldots, (S^K, x^K) \in \mathcal{B}_C$,

$$[g(S^k, x^k) = x^{k-1} \text{ for all } k \in \{1, \ldots, K\} \text{ and } g(S^0, x^0) = x^K] \Rightarrow x^K \notin C(S^0).$$

Rather than merely the existence of a function in $\mathcal{G}_C$ with this property, a necessary and sufficient condition for full rationalizability requires *all* functions in $\mathcal{G}_C$ to satisfy $\overline{\mathbf{D}}$; again, this is an immediate consequence of the more demanding nature of M-rationalizability by an ordering and its equivalence to G-rationalizability.

$\overline{\mathbf{D}}$-axiom: For all $g \in \mathcal{G}_C$, g satisfies $\overline{\mathbf{D}}$.

In the proof of our next characterization result, we make use of Richter's (1966) characterization of full rationalizability.[2] Richter (1966) shows that the *congruence* axiom is necessary and sufficient for greatest-element rationalizability by an ordering. In our setting, congruence can be expressed as follows.

Congruence: For all $K \in \mathbb{N}$ and for all $(S^0, x^0), \ldots, (S^K, x^K) \in \mathcal{B}_C$,

$$[x^{k-1} \in C(S^k) \text{ for all } k \in \{1, \ldots, K\} \text{ and } x^K \in C(S^0)] \Rightarrow x^0 \in C(S^0).$$

This property can be used to prove our characterization of full rationalizability.

Theorem 6 *C satisfies **CS-M** if and only if C satisfies the $\overline{\mathbf{D}}$-axiom.*

Proof. By Richter's (1966) result and the observation that **RCT-G** is equivalent to the axiom in the theorem statement, it is sufficient to establish the equivalence of the congruence axiom and the $\overline{\mathbf{D}}$-axiom.

Step 1 We first prove that the congruence axiom implies the $\overline{\mathbf{D}}$-axiom. Suppose C satisfies the congruence axiom. Let $g \in \mathcal{G}_C$ and suppose that $K \in \mathbb{N}$ and $(S^0, x^0), \ldots, (S^K, x^K) \in \mathcal{B}_C$ are such that $g(S^k, x^k) = x^{k-1}$ for all $k \in \{1, \ldots, K\}$ and $g(S^0, x^0) = x^K$. Because $g \in \mathcal{G}_C$, it follows that $x^{k-1} \in C(S^k)$ for all $k \in \{1, \ldots, K\}$ and $x^K \in C(S^0)$. By the congruence axiom, $x^0 \in C(S^0)$ and, thus, g satisfies $\overline{\mathbf{D}}$.

Step 2 To prove the reverse implication, suppose that C satisfies the $\overline{\mathbf{D}}$-axiom. That is, every $g \in \mathcal{G}_C$ satisfies $\overline{\mathbf{D}}$. Suppose $K \in \mathbb{N}$ and $(S^0, x^0), \ldots, (S^K, x^K) \in \mathcal{B}_C$ are such that $x^{k-1} \in C(S^k)$ for all $k \in \{1, \ldots, K\}$ and $x^K \in C(S^0)$. Let $g \in \mathcal{G}_C$ be such that $g(S^k, x^k) = x^{k-1}$ for all $k \in \{1, \ldots, K\}$ and $g(S^0, x^0) = x^K$. Clearly, such a function g exists. By the $\overline{\mathbf{D}}$-axiom, g satisfies $\overline{\mathbf{D}}$. Therefore, it follows that $x^0 \in C(S^0)$ and hence the congruence axiom is satisfied. ∎

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[2]See Hansson (1968) and Suzumura (1977; 1983, Chapter 2, Appendix A) for two alternative characterizations.

6   Single-Valued Choice Functions

In the case of single-valued choice functions (that is, choice functions such that $|C(S)| = 1$ for all $S \in \Sigma$, where $|C(S)|$ is the cardinality of $C(S)$), our characterization of full rationalizability can be simplified considerably. Note, however, that this is not the case for weaker notions of M-rationalizability. This difference is due to the observation that weak M-rationalizabilty merely requires every feasible element y, that is not chosen in a set S, to be dominated (in the sense of strict preference) by an element x in S, where x need not be chosen itself. Thus, the assumption that $C(S)$ contains a single element only does not simplify matters as far as the identification of necessary and sufficient conditions for M-rationalizable choice is concerned.

To characterize full rationalizability in the single-valued setting, our first modification is to restrict attention to those functions f that map into $C(S)$; due to the full-rationalizability assumption, any element that is not chosen must be dominated by some element in the chosen set, given that the rationalization is an ordering. We define

$$\mathcal{H}_C = \{f \colon \mathcal{A}_C \to X \mid f(S, y) \in C(S) \text{ for all } (S, y) \in \mathcal{A}_C\}.$$

Our necessary and sufficient condition is the following $\overline{\mathbf{E}}$-axiom.

$\overline{\mathbf{E}}$-**axiom:** If $\mathcal{A}_C \neq \emptyset$, then there exists $f \in \mathcal{H}_C$ satisfying $\overline{\mathbf{A}}$.

We now prove our final characterization result. As in the previous section, we make use of Richter's (1966) result and show that the congruence axiom is equivalent to the $\overline{\mathbf{E}}$-axiom if C is single-valued.

Theorem 7 *Suppose C is single-valued. C satisfies* ***CS-M*** *if and only if C satisfies the* $\overline{\mathbf{E}}$-*axiom.*

Proof. We assume that C is single-valued and prove the equivalence of congruence and the $\overline{\mathbf{E}}$-axiom.

Step 1 Suppose C satisfies the congruence axiom. If $\mathcal{A}_C = \emptyset$, the result is immediate. Now suppose $\mathcal{A}_C \neq \emptyset$. We define a function $f \in \mathcal{H}_C$ and show that, for that function, $\overline{\mathbf{A}}$ is satisfied. For all $(S, y) \in \mathcal{A}_C$ and for all $x \in X$, let $f(S, y) = x$ if and only if $x \in C(S)$. Because C is single-valued, f is well-defined and unique. To prove that the $\overline{\mathbf{E}}$-axiom is satisfied, suppose $K \in \mathbb{N}$, $(S^1, x^1), \ldots, (S^K, x^K) \in \mathcal{A}_C$, $S^0 \in \Sigma$ and $x^0 \in S^0$ are such that $f(S^k, x^k) = x^{k-1}$ for all $k \in \{1, \ldots, K\}$.

By way of contradiction, suppose $x^K \in C(S^0)$. Because $\mathcal{A}_C \subseteq \mathcal{B}_C$, it follows that $(S^1, x^1), \ldots, (S^K, x^K) \in \mathcal{B}_C$. Furthermore, by assumption, $(S^0, x^0) \in \mathcal{B}_C$. The congruence axiom implies $x^0 \in C(S^0)$ and, because C is single-valued, we must have $x^0 = x^K$. Therefore, $x^K = x^0 = f(S^1, x^1)$ and, if $K > 1$, $x^{k-1} = f(S^k, x^k)$ for all $k \in \{2, \ldots, K\}$. By definition of f, this implies $x^K \in C(S^1)$ and $x^1 \in S^1 \setminus C(S^1)$ and, furthermore, $x^{k-1} \in C(S^k)$ and $x^k \in S^k \setminus C(S^k)$ for all $k \in \{2, \ldots, K\}$. This is a contradiction to the congruence axiom, which completes Step 1.

Step 2 Suppose C satisfies the $\overline{\mathbf{E}}$-axiom. To show that the congruence axiom is satisfied, let $K \in \mathbb{N}$ and $(S^0, x^0), \ldots, (S^k, x^K) \in \mathcal{B}_C$ be such that $x^{k-1} \in C(S^k)$ for all $k \in \{1, \ldots, K\}$ and $x^K \in C(S^0)$. If $x^0 = x^K$, the result is immediate.

Now consider the case $x^0 \neq x^K$. Suppose there exist $k, j \in \{0, \ldots, K\}$ such that $k \neq j$ and $x^k = x^j$. Without loss of generality, suppose that $k < j$. If $j = K$, it follows that $x^k = x^K \in C(S^0)$; if $j < K$, we obtain $x^k = x^j \in C(S^{j+1})$. In either case, a revealed preference chain of a lower order is obtained because the elements $x^k, \ldots, x^{j-1}$ can be omitted. Because K is finite and $x^0 \neq x^K$, this argument can be repeated sufficiently many times to obtain a revealed preference chain of some order where all elements are distinct. Thus, we can without loss of generality assume that $x^k \neq x^j$ for all $k, j \in \{0, \ldots, K\}$ such that $k \neq j$.

Because all elements in this revealed preference chain are distinct, single-valuedness implies that $x^k \in S^k \setminus C(S^k)$ for all $k \in \{1, \ldots, K\}$ and, therefore, $(S^0, x^0), \ldots, (S^K, x^K) \in \mathcal{A}_C$. This implies $\mathcal{A}_C \neq \emptyset$ and, by the $\overline{\mathbf{E}}$-axiom, there exists $f \in \mathcal{H}_C$ satisfying $\overline{\mathbf{A}}$. Because C is single-valued and $f(S, y) \in C(S)$ for all $(S, y) \in \mathcal{A}_C$ by definition, it follows that $f(S^k, x^k) = x^{k-1}$ for all $k \in \{1, \ldots, K\}$. Because $x^K \in C(S^0)$, we obtain a contradiction to $\overline{\mathbf{A}}$. Hence the case $x^0 \neq x^K$ cannot occur and the proof is complete. ∎

7   Concluding Remarks

The classical notion of rational choice as purposive behavior has been extensively explored in the literature by interpreting purposive behavior as 'optimizing behavior' with respect to some underlying preference relation or utility function. This paper analyzes an alternative interpretation of purposive behavior as maximizing behavior with respect to some underlying preference relation that need not be reflexive and complete. Within the class of choice functions which are maximal-element rationalizable, we identify several important sub-classes and characterize them in terms of intuitive axioms. No restrictions on the domain of a choice function are imposed other than the non-emptiness of the do-

main, and the decisiveness of choice. As a by-product of our analysis of maximal-element rationalizability, we shed further light on a problem left open in the classical study by Richter (1971) on greatest-element rationalizability.

Except for the case of full rationalizability (rationalizability by an ordering), our characterizations involve existential clauses. This is sometimes seen as a shortcoming but it seems to us that this objection, by itself, does not stand on solid ground: there is nothing inherently undesirable in an axiom involving existential clauses. If the argument is that existential clauses are difficult to verify in practice, this is easily countered by the observation that universal quantifiers are no easier to check algorithmically (at least, in the case of existential clauses, a search algorithm can terminate once one object with the desired property can be found). Thus, in this respect, our conditions compare rather favorably with those that are required for many forms of greatest-element rationalizability where universal quantifiers play a dominant role.

We suspect that a major reason behind the reluctance to accept existential clauses in the context of rational choice may be that conditions involving existential requirements are seen as being 'too close' to the rationalizability property itself because the desired property is expressed in terms of the existence of a rationalization. This is (except for obvious cases) a matter of judgement, of course. Our view is that the combinations of the axioms employed in the characterizations of the weak forms of rationalizability represent an interesting and insightful way of separating the properties involved in maximal-element rationalizability. Furthermore, the axioms we use appear to be rather clear and the roles they play in the respective results have very intuitive interpretations. By definition of maximal-element rationalizability, existential clauses appear naturally and it is therefore not surprising that this feature is reflected in our conditions as well. Finally, we should observe that the mathematical structures encountered are similar to those appearing in dimension theory and, consequently, closely related complexities cannot but arise. In fact, existential clauses appear in many of the characterization results in that area; see, for example, Dushnik and Miller (1941).[3]

In concluding this paper, some remarks on further problems to be explored are in order. Because we do not impose any restrictions on the domain of a choice function (other than non-emptiness), our results are extremely general. As a result, our theorems can be of relevance in whatever context of rational choice as purposive behavior we may care to specify, which is an obvious merit of our general approach. Note, however, that our

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[3]Dimension theory addresses the question of how many orderings are required to express a quasi-ordering as the intersection of these orderings.

approach may overlook some meaningful further directions to explore by being insensitive to the structural properties of the domain which make perfect sense in the specific contexts on which we are focussing. Two representative examples may be worthwhile to mention. The first structural property of the domain is *finite additivity* stating that, for any two sets in the domain, their set-theoretical union is also a member of the domain. The second structural property of the domain is *coveredness:* for any two sets in the domain, there exists a member of the domain which contains the set-theoretical union of the two sets we have started from. Note that the former structural property of the domain is not satisfied by the Samuelson-Houthakker domain which consists of the budget sets in a commodity space, whereas the latter structural property of the domain is. Indeed, the second property can be construed as a generalization of the first; it is also a property which is satisfied by the Arrow-Sen domain with a finite universal set. We suggest that the exploration of the theory of rational choice under one or the other of these restrictions (or yet others not mentioned here) is a worthwile direction of future research in the specific context of our discourse.

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