Arrovian Aggregation in Economic Environments: How Much Should We Know About Indifference Surfaces?*

Marc Fleurbaey† Kotaro Suzumura‡ Koichi Tadenuma§

November 2002
(First version: July 2001)

*The authors thank A. Leroux for a stimulating discussion, A. Trannoy for comments, and participants at seminars in Cergy, Rochester and Hitotsubashi, and the 5th International Meeting of the Society for Social Choice and Welfare in Alicante. Financial support from the Ministry of Education, Science and Culture of Japan through Grant-in-Aid for Scientific Research No.10045010 ("Economic Institutions and Social Norms: Evolution and Transformation") is gratefully acknowledged.

†CATT, THEMA, IDEP, University of Pau, Av. du Doyen Poplawski, 64000 PAU, France. Email: Marc.Fleurbaey@univ-pau.fr

‡Institute of Economic Research, Hitotsubashi University, Kunitachi, Tokyo 186-8603, Japan. Email: suzumura@ier.hit-u.ac.jp

§Corresponding author. Faculty of Economics, Hitotsubashi University, Kunitachi, Tokyo 186-8601, Japan. Phone: +(81) 42-580-8603. Fax: +(81) 42-580-8748. Email: tadenuma@econ.hit-u.ac.jp

Abstract

Arrow's celebrated theorem of social choice shows that the aggregation of individual preferences into a social ordering cannot make the ranking of any pair of alternatives depend only on individual preferences over that pair, unless the fundamental weak Pareto and non-dictatorship principles are violated. In the standard model of division of commodities, we investigate how much information about indifference hypersurfaces is needed to construct social ordering functions satisfying the weak Pareto principle and anonymity. We show that local information such as marginal rates of substitution or the shapes "within the Edgeworth box" is not enough, and knowledge of substantially non-local information is necessary.

Key words: social choice, preference aggregation, information, independence of irrelevant alternatives, indifference curves.

Journal of Economic Literature Classification Numbers: D63, D71.

1 Introduction

From Arrow's celebrated theorem of social choice, it is well known that the aggregation of individual preferences into a social ordering cannot make the social ranking of any pair of alternatives depend only on individual preferences over that pair (this is the famous axiom of Independence of Irrelevant Alternatives). Or, more precisely, it cannot do so without trespassing basic requirements of unanimity (the Pareto principle) and anonymity (even in the very weak version of non-dictatorship). This raises the following question: What additional information about preferences would be needed in order to make aggregation of preferences possible, and compatible with the basic requirements of unanimity and anonymity?

In the last decades, the literature on social choice has explored several paths and gave interesting answers to this question. The main avenue of research has been, after Sen [18] and d'Aspremont and Gevers [7], the introduction of information about utilities, and it has been shown that the classical social welfare functions, and less classical ones, could be obtained with the Arrovian axiomatic method by letting the social preferences take account of specific kinds of utility information.

In this paper, we focus on the introduction of additional information about preferences that is not of the utility sort. In other words, we retain a framework with purely ordinal and interpersonally non-comparable preferences. The kind of additional information that we study is about the shapes of indifference curves, and we ask how much one needs to know about indifference curves so as to be able to aggregate individual preferences while respecting the unanimity and anonymity requirements. The introduction of this additional information is formulated in terms of weakening Arrow's axiom of independence of irrelevant alternatives.

The model adopted here is an economic model, namely, the canonical model of division of infinitely divisible commodities among a finite set of agents. We chose to study an economic model rather than the abstract model that is now commonly used in the theory of social choice[1] for two reasons. First, it allows a more fine-grained analysis of information about preferences, because it makes it sensible to talk about marginal rates of substitution and other local notions about indifference curves. Second, in an economic

[1]Recollect, however, that Arrow's initial presentations [1, 2] dealt with this economic model of division of commodities.

model preferences are naturally restricted, and by considering a restricted domain we can hope to obtain positive results with less information than under unrestricted domain.

Our first extension of informational basis is to take account of marginal rates of substitution. It turns out that such infinitesimally local information would not be enough to escape from dictatorship, and we establish an extension of Arrow's theorem. Then, it is natural to take account of the portions of indifference curves in some finitely sized neighborhoods of the allocations. Based on this additional information, we can construct a non-dictatorial aggregation rule or *social ordering function*, but still anonymity cannot be attained.

The second direction of extending informational basis focuses on indifference curves "within the Edgeworth box". More precisely, for any two allocations, we define the smallest vector of total resources that makes both allocations feasible, and take the portion of the indifference curve through each allocation in the region below the vector. The introduction of this kind of information, however, does not help us avoid dictatorship.

The third avenue relies on some fixed monotone path from the origin in the consumption space, and focuses on the points of indifference curves that belong to this path. The idea of referring to such a monotone path is due to Pazner and Schmeidler [16], and may be justified if the path contains relevant benchmark bundles. Making use of this additional information, and following Pazner and Schmeidler's [16] contribution, we can construct a unanimous and anonymous social ordering function.

Our final, the largest, extension of informational basis is to take whole indifference curves. Given the above result, a unanimous and anonymous social ordering function can be constructed on this informational basis.

The motivation for our research draws on many strands of recent and less recent literature. Attempts to construct social ordering functions and similar objects embodying unanimity and equity requirements were made by Suzumura [19, 20] and Tadenuma [21]. The idea that information about whole indifference curves is sufficient, hinted at by Pazner and Schmeidler [16] and Maniquet [14], was made more precise in Pazner [15] and was revived by Bossert, Fleurbaey and Van de gaer [4] and Fleurbaey and Maniquet [8, 9] who were able to construct nicely behaved social ordering functions on this basis. Campbell and Kelly [5] recently studied essentially the same issue in an abstract model of social choice, and showed that limited information about preferences may be enough. However, their model does not have the rich structure of economic environments, and they focus only on non-dictatorship and do not study how much information is needed for the stronger requirement of anonymity.

The paper is organized as follows. The next section introduces the framework and the main notions. The results are presented in Section 3. Section 4 concludes. The appendix contains some proofs.

2 The Model and Axioms

2.1 The model

The *population* is fixed. Let $N = \{1, ..., n\}$ be the set of *agents* where $2 \leq n < \infty$. There are ℓ *goods* indexed by $k = 1, ..., \ell$ where $2 \leq \ell < \infty$. Agent i's *consumption bundle* is a vector $x_i = (x_{i1}, ..., x_{i\ell})$. An *allocation* is denoted $x = (x_1, ..., x_n)$. The *set of allocations* is $\mathbb{R}_+^{n\ell}$. The set of allocations such that no individual bundle x_i is equal to the zero vector is denoted X.

A *preordering* is a reflexive and transitive binary relation. Agent i's *preferences* are described by a complete preordering R_i (strict preference P_i, indifference I_i) on $\mathbb{R}_+^{\ell}$. A *profile of preferences* is denoted $\mathbf{R} = (R_1, ..., R_n)$. Let $\mathcal{R}$ be the set of continuous, convex, and strictly monotonic preferences over $\mathbb{R}_+^{\ell}$.

A *social ordering function* (SOF) is a mapping $\bar{R}$ defined on $\mathcal{R}^n$, such that for all $\mathbf{R} \in \mathcal{R}^n$, $\bar{R}(\mathbf{R})$ is a complete preordering on the set of allocations $\mathbb{R}_+^{n\ell}$. Let $\bar{P}(\mathbf{R})$ (resp. $\bar{I}(\mathbf{R})$) denote the strict preference (resp. indifference) relation associated to $\bar{R}(\mathbf{R})$.

Let π be a bijection on N. For all $x \in \mathbb{R}_+^{n\ell}$, define $\pi(x) = (x'_1, ..., x'_n) \in \mathbb{R}_+^{n\ell}$ by $x'_i = x_{\pi(i)}$ for all $i \in N$, and for all $\mathbf{R} \in \mathcal{R}^n$, define $\pi(\mathbf{R}) = (R'_1, ..., R'_n) \in \mathcal{R}^n$ by $R'_i = R_{\pi(i)}$ for all $i \in N$. Let Π be the set of all bijections on N. The basic requirements of unanimity and anonymity on which we focus in this paper are the following.

Weak Pareto: $\forall \mathbf{R} \in \mathcal{R}^n$, $\forall x, y \in \mathbb{R}_+^{n\ell}$, if $\forall i \in N, x_i P_i y_i$, then $x \bar{P}(\mathbf{R}) y$.

Anonymity: $\forall \mathbf{R} \in \mathcal{R}^n, \forall x, y \in \mathbb{R}_+^{n\ell}, \forall \pi \in \Pi$:

$$x \bar{R}(\mathbf{R}) y \Leftrightarrow \pi(x)\ \bar{R}(\pi(\mathbf{R}))\ \pi(y).$$

Concerning the non-dictatorship form of anonymity, we only define here what dictatorship means, for convenience. Notice that it has to do only with allocations in X, that is, without the zero bundle for any agent.

Dictatorial SOF: A SOF $\bar{R}$ is dictatorial if there exists $i_0 \in N$ such that:

$$\forall \mathbf{R} \in \mathcal{R}^n, \forall x, y \in X : x_{i_0} P_{i_0} y_{i_0} \Rightarrow x \bar{P}(\mathbf{R}) y.$$

2.2 Variants of Independence of Irrelevant Alternatives

The traditional, Arrovian, version of Independence of Irrelevant Alternatives is:

Independence of Irrelevant Alternatives (IIA): $\forall \mathbf{R}, \mathbf{R}' \in \mathcal{R}^n$, $\forall x, y \in \mathbb{R}_+^{n\ell}$, if $\forall i \in N$, R_i and R_i' agree on $\{x_i, y_i\}$, then $\bar{R}(\mathbf{R})$ and $\bar{R}(\mathbf{R}')$ agree on $\{x, y\}$.

It is possible to weaken IIA by strengthening the premise. This amounts to allowing the SOF to make use of more information when ranking each pair of allocations.

First, we consider making use of marginal rates of substitution. Economists are used to focus on marginal rates of substitution when assessing the efficiency of an allocation, especially under convexity, since for convex preferences the marginal rates of substitution determine the half space in which the upper contour set lies. Moreover, for efficient allocations, the shadow prices enable one to compute the relative implicit income shares of different agents, thereby potentially providing a relevant measure of inequalities in the distribution of resources. Therefore, taking account of marginal rates of substitution is a natural extension of the informational basis of social choice in economic environments.

Let $C(x_i, R_i)$ denote the cone of price vectors that support the upper contour set for R_i at x_i :

$$C(x_i, R_i) = \{p \in \mathbb{R}^\ell | \forall y \in \mathbb{R}_+^\ell, py = px_i \Rightarrow x_i R_i y\}.$$

When preferences R_i are strictly monotonic, one has $C(x_i, R_i) \subset \mathbb{R}_{++}^\ell$ whenever $x_i \gg 0$.

IIA except Marginal Rates of Substitution (IIA-MRS): $\forall \mathbf{R}, \mathbf{R}' \in \mathcal{R}^n$, $\forall x, y \in \mathbb{R}_+^{n\ell}$, if $\forall i \in N$, R_i and R_i' agree on $\{x_i, y_i\}$, and

$$\begin{aligned} C(x_i, R_i) &= C(x_i, R_i'), \\ C(y_i, R_i) &= C(y_i, R_i'), \end{aligned}$$

then $\bar{R}(\mathbf{R})$ and $\bar{R}(\mathbf{R}')$ agree on $\{x, y\}$.

Marginal rates of substitution give an infinitesimally local piece of information about indifference hypersurfaces at given allocations. A natural extension of the informational basis would be to take account of the indifference hypersurfaces in some finitely sized neighborhoods of the two allocations. Define, for any given real number $\varepsilon > 0$,

$$B_\varepsilon(x_i) = \{v \in \mathbb{R}^\ell_+ | \max_{k \in \{1,\ldots,\ell\}} |x_{ik} - v_k| \leq \varepsilon\}$$

Define

$$I(x_i, R_i) = \{z \in \mathbb{R}^\ell_+ \mid z \; I_i \; x_i\}.$$

The set $I(x_i, R_i)$ is called the *indifference set* at x_i for R_i.

The next axiom of SOFs is defined for a given $\varepsilon > 0$. Notice that the larger (the smaller) is the value of ε, the weaker (the stronger) the condition becomes.

IIA except Indifference Sets in ε-Neighborhoods (IIA-ISεN): Let $\varepsilon > 0$ be given. $\forall \mathbf{R}, \mathbf{R}' \in \mathcal{R}^n$, $\forall x, y \in \mathbb{R}^{n\ell}_+$, if $\forall i \in N$, R_i and R'_i agree on $\{x_i, y_i\}$, and

$$\begin{aligned} I(x_i, R_i) \cap B_\varepsilon(x_i) &= I(x_i, R'_i) \cap B_\varepsilon(x_i), \\ I(y_i, R_i) \cap B_\varepsilon(y_i) &= I(y_i, R'_i) \cap B_\varepsilon(y_i), \end{aligned}$$

then $\bar{R}(\mathbf{R})$ and $\bar{R}(\mathbf{R}')$ agree on $\{x, y\}$.

The second type of extension of informational basis is to focus on the portions of indifference sets which lie "within the Edgeworth box". However, when considering any pair of allocations, the two allocations may need different amounts of total resources to be feasible. Therefore we need to introduce the following notions. For each good $k \in \{1, ..., \ell\}$, define

$$\omega_k(x, y) \equiv \max\{\sum_{i \in N} x_{ik}, \sum_{i \in N} y_{ik}\}.$$

Let $\omega(x, y) = (\omega_1(x, y), ..., \omega_\ell(x, y))$. The vector $\omega(x, y) \in \mathbb{R}^\ell_+$ represents the smallest amount of total resources that makes two allocations x and y feasible. Then, define

$$\Omega(x, y) = \left\{z \in \mathbb{R}^\ell_+ \mid z \leq \omega(x, y)\right\}.$$

The set $\Omega(x,y) \subset \mathbb{R}^\ell_+$ is the set of consumption bundles that are feasible with $\omega(x,y)$. The following axiom captures the idea that the ranking of two allocations should depend only on the indifference hypersurfaces over the region satisfying the feasibility constraint.

IIA except Indifference Sets over Feasible Allocations (IIA-ISFA): $\forall \mathbf{R}, \mathbf{R}' \in \mathcal{R}^n$, $\forall x, y \in \mathbb{R}^{n\ell}_+$, if $\forall i \in N$,

$$\begin{aligned} I(x_i, R_i) \cap \Omega(x,y) &= I(x_i, R'_i) \cap \Omega(x,y), \\ I(y_i, R_i) \cap \Omega(x,y) &= I(y_i, R'_i) \cap \Omega(x,y), \end{aligned}$$

then $\bar{R}(\mathbf{R})$ and $\bar{R}(\mathbf{R}')$ agree on $\{x,y\}$.

It will actually be worth considering weaker variants of this axiom, which rely on radial expansions of the set $\Omega(x,y)$. For any set $Y \subset \mathbb{R}^\ell$ and any $\lambda \geq 1$, define

$$\lambda Y = \{q \in \mathbb{R}^\ell | \lambda^{-1} q \in Y\}.$$

The next axiom is defined for a given $\lambda \geq 1$. The larger is the value of λ, the weaker the axiom becomes.

IIA except Indifference Sets over λ-Expanded Feasible Allocations (IIA-ISλEFA): $\forall \mathbf{R}, \mathbf{R}' \in \mathcal{R}^n$, $\forall x, y \in \mathbb{R}^{n\ell}_+$, if $\forall i \in N$,

$$\begin{aligned} I(x_i, R_i) \cap \lambda\Omega(x,y) &= I(x_i, R'_i) \cap \lambda\Omega(x,y), \\ I(y_i, R_i) \cap \lambda\Omega(x,y) &= I(y_i, R'_i) \cap \lambda\Omega(x,y), \end{aligned}$$

then $\bar{R}(\mathbf{R})$ and $\bar{R}(\mathbf{R}')$ agree on $\{x,y\}$.

Combining ε-neighborhoods of two allocations x, y and a radial expansion of $\Omega(x,y)$ as the informational basis, we have the following axiom. Let $\varepsilon > 0$ and $\lambda \geq 1$ be given.

IIA except Indifference Sets in ε-Neighborhoods and over λ-Expanded Feasible Allocations (IIA-IS[εNλEFA]): $\forall \mathbf{R}, \mathbf{R}' \in \mathcal{R}^n$, $\forall x, y \in \mathbb{R}^{n\ell}_+$, if $\forall i \in N$,

$$\begin{aligned} I(x_i, R_i) \cap [B_\varepsilon(x_i) \cup \lambda\Omega(x,y)] &= I(x_i, R'_i) \cap [B_\varepsilon(x_i) \cup \lambda\Omega(x,y)] \\ I(y_i, R_i) \cap [B_\varepsilon(y_i) \cup \lambda\Omega(x,y)] &= I(y_i, R'_i) \cap [B_\varepsilon(y_i) \cup \lambda\Omega(x,y)], \end{aligned}$$

then $\bar{R}(\mathbf{R})$ and $\bar{R}(\mathbf{R}')$ agree on $\{x,y\}$.

The third way of extending information about indifference sets is to rely on a path

$$\Lambda_{\omega_0} = \{\lambda\omega_0 \in \mathbb{R}^{\ell}_{++} \mid \lambda \in \mathbb{R}_+\},$$

where $\omega_0 \in \mathbb{R}^{\ell}_{++}$ is fixed, and to focus on the part of the indifference sets which belongs to this path. The idea of referring to such a path is due to Pazner and Schmeidler [16], and may be justified if the path contains relevant benchmark bundles. The choice of ω_0 is not discussed here, but it need not be arbitrary. For instance, one may imagine that it could reflect an appropriate equity notion, or it could be the bundle of total resource availability.

IIA except Indifference Sets on Path ω_0 (IIA-ISPω_0): $\forall \mathbf{R}, \mathbf{R}' \in \mathcal{R}^n$, $\forall x, y \in \mathbb{R}^{n\ell}_+$, if $\forall i \in N$,

$$\begin{aligned} I(x_i, R_i) \cap \Lambda_{\omega_0} &= I(x_i, R_i') \cap \Lambda_{\omega_0}, \\ I(y_i, R_i) \cap \Lambda_{\omega_0} &= I(y_i, R_i') \cap \Lambda_{\omega_0}, \end{aligned}$$

then $\bar{R}(\mathbf{R})$ and $\bar{R}(\mathbf{R}')$ agree on $\{x, y\}$.

The final extension of informational basis that we consider is to introduce whole indifference hypersurfaces. This condition was already introduced and studied by Hansson [11] in the abstract model of social choice, who showed that the Borda rule, which does not satisfy the Arrow IIA condition, satisfies this constrained variant thereof. Pazner [15] also proposed it, in a study of social choice in economic environments.

IIA except Whole Indifference Sets (IIA-WIS): $\forall \mathbf{R}, \mathbf{R}' \in \mathcal{R}^n$, $\forall x, y \in \mathbb{R}^{n\ell}_+$, if $\forall i \in N$,

$$\begin{aligned} I(x_i, R_i) &= I(x_i, R_i'), \\ I(y_i, R_i) &= I(y_i, R_i'), \end{aligned}$$

then $\bar{R}(\mathbf{R})$ and $\bar{R}(\mathbf{R}')$ agree on $\{x, y\}$.

Lemma 1 *Let $\varepsilon > 0$ and $\lambda \geq 1$ be given.*

$$\begin{array}{ccccccccc} & & \textit{IIA-MRS} & \rightarrow & \textit{IIA-IS}\varepsilon N & & & & \\ & \nearrow & & & & \searrow & & & \\ \textit{IIA} & \rightarrow & \textit{IIA-ISFA} & \rightarrow & \textit{IIA-IS}\lambda\textit{EFA} & \rightarrow & \textit{IIA-IS}[\varepsilon N\lambda \textit{EFA}] & \rightarrow & \textit{IIA-WIS} \\ & \searrow & & & & & & \nearrow & \\ & & & & \textit{IIA-ISP}\omega_0 & & & & \end{array}$$

3 How large portions of indifference surfaces do we have to know?

Let us first recall the formulation of Arrow's theorem for this model (Bordes and Le Breton [3]).

Proposition 1 *If a SOF $\bar{R}$ satisfies Weak Pareto and IIA, then it is dictatorial.*

It turns out, unfortunately, that introducing information about marginal rates of substitution, in addition to pairwise preferences, does not make room for the existence of satisfactory SOFs. More formally, weakening IIA into IIA-MRS does not alter the dictatorship conclusion of Arrow's theorem.

Proposition 2 *If a SOF $\bar{R}$ satisfies Weak Pareto and IIA-MRS, then it is dictatorial.*

The proof of this Proposition is long and is relegated to the appendix, but here we sketch the main line of the proof. Since IIA implies IIA-MRS, Proposition 2 is a generalization of the theorem by Bordes and Le Breton [3, Theorem 3]. An essential idea of the proofs of Arrow-like theorems in economic environments (Kalai, Muller and Satterthwaite [13], Bordes and Le Breton [3], and others) is as follows: First, we find a "free triple", that is, three allocations for which any ranking is possible in each individual's preferences satisfying the standard assumptions in economics. By applying Arrow's theorem for these three allocations, it can be shown that there exists a "local dictator" for each free triple. Then, we "connect" free triples in a suitable way to show that these local dictators must be the same individual.

Turning to IIA-MRS, notice first that for each free triple, IIA-MRS works just as IIA only in the class of preference profiles for which all individuals' marginal rates of substitution at the three allocations are the same, and satisfy certain "supporting conditions". Invoking Arrow's theorem, we can only show that there exists a "local dictator" for each free triple in this much restricted class of preference profiles (Lemmas 2 and 3). The difficulty in the proof of Proposition 2 lies in extending "local dictatorship" over the class of all preference profiles. This requires much work to do. See Lemmas 4 and 5 in the Appendix.

Inada [12] also considered marginal rates of substitution in an IIA-like axiom, but the difference from our work is that he looked for a local aggregator of preferences, namely a mapping defining a social marginal rate of substitution between goods and individuals, on the basis of individual marginal rates of substitution. Hence, Inada requires that, for each allocation, social preferences *in an infinitely small neighborhood of the allocation* should not change whenever every agent's marginal rates of substitution at the allocation remain the same. By contrast, our IIA-MRS requires that, for each pair of allocations, social preferences *over that pair* should not change whenever every agent's marginal rates of substitution at each of the two allocations remain the same. There is no logical relation between Inada's axiom and ours.

The next proposition shows that as soon as one switches from IIA-MRS to IIA-ISεN, the dictatorship result is avoided, even if ε is arbitrarily small. However, it remains impossible to achieve Anonymity, even for an arbitrarily large ε.

Proposition 3 *Let $\varepsilon > 0$ be given. There exists a SOF that satisfies Weak Pareto, IIA-ISεN, and is not dictatorial. However, there exists no SOF that satisfies Weak Pareto, IIA-ISεN and Anonymity.*

Proof. The impossibility part is derived directly from Proposition 5 below, and here we omit the proof.

We prove the possibility part. Define $\bar{R}$ as follows: $x\bar{R}(\mathbf{R})y$ if either $x_1R_1y_1$ and $[I(x_1, R_1) \nsubseteq B_\varepsilon(0)$ or $I(y_1, R_1) \nsubseteq B_\varepsilon(0)]$, or $x_2R_2y_2$ and $[I(x_1, R_1) \subseteq B_\varepsilon(0)$ and $I(y_1, R_1) \subseteq B_\varepsilon(0)]$. For brevity, let $\Gamma(v)$ denote $[I(v, R_1) \subseteq B_\varepsilon(0)]$. Weak Pareto and the absence of dictator are straightforwardly satisfied. IIA-ISεN is also satisfied because when $\Gamma(x_1)$ and $\Gamma(y_1)$ hold, we have $B_\varepsilon(0) \subseteq B_\varepsilon(x_1) \cap B_\varepsilon(y_1)$, and therefore $\Gamma(x_1)$ and $\Gamma(y_1)$ remain true if the indifference surfaces are kept fixed on $B_\varepsilon(x_1)$ and $B_\varepsilon(y_1)$. It remains to check transitivity of $\bar{R}(\mathbf{R})$. First, note the following property: If $\Gamma(v)$ holds and vR_1v', then $\Gamma(v')$ also holds. Assume that there exist $x, y, z \in \mathbb{R}^{n\ell}_+$ such that $x\bar{R}(\mathbf{R})y\bar{R}(\mathbf{R})z\bar{P}(\mathbf{R})x$. If $\Gamma(x_1)$, $\Gamma(y_1)$ and $\Gamma(z_1)$ all hold, this is impossible because one should have $x_2R_2y_2R_2z_2P_2x_2$. If only one of the three conditions $\Gamma(x_1)$, $\Gamma(y_1)$, $\Gamma(z_1)$ is satisfied, it is similarly impossible because one should have $x_1R_1y_1R_1z_1P_1x_1$. Assume $\Gamma(x_1)$ and $\Gamma(y_1)$ hold, but not $\Gamma(z_1)$. Then, $y\bar{R}(\mathbf{R})z\bar{P}(\mathbf{R})x$ requires $y_1R_1z_1P_1x_1$, which implies $\Gamma(z_1)$, a contradiction. Assume $\Gamma(x_1)$ and $\Gamma(z_1)$ hold, but not $\Gamma(y_1)$. Then, $x\bar{R}(\mathbf{R})y\bar{R}(\mathbf{R})z$ requires $x_1R_1y_1R_1z_1$, which implies $\Gamma(y_1)$, a contradiction. Assume $\Gamma(y_1)$ and $\Gamma(z_1)$ hold, but not $\Gamma(x_1)$. Then, $z\bar{P}(\mathbf{R})x\bar{R}(\mathbf{R})y$ requires $z_1P_1x_1R_1y_1$, which implies $\Gamma(x_1)$, a contradiction. ■

Let us next consider the second direction of extending informational basis, focusing on indifference curves "in the Edgeworth box". The introduction of such information about indifference curves, however, cannot help us avoid a dictatorial SOF.

Proposition 4 *If a SOF satisfies Weak Pareto and IIA-ISFA, then it is dictatorial.*

The proof relies on the following lemmas. First, we define a weak form of IIA:

Weak Independence of Irrelevant Alternatives (WIIA): $\forall \mathbf{R}, \mathbf{R}' \in \mathcal{R}^n$, $\forall x, y \in X$, if $\forall i \in N$, R_i and R'_i agree on $\{x_i, y_i\}$, and for no i, $x_i I_i y_i$, then $\bar{R}(\mathbf{R})$ and $\bar{R}(\mathbf{R}')$ agree on $\{x, y\}$.

A key lemma to prove Proposition 4 is the following:

Lemma 2 *If a SOF $\bar{R}$ satisfies Weak Pareto and IIA-ISFA, then it satisfies WIIA.*

The proof of this lemma is long and relegated in the appendix. We also define a weak form of dictatorship:

Quasi-Dictatorial SOF: A SOF $\bar{R}$ is quasi-dictatorial if there exists $i_0 \in N$ such that:

$$\forall \mathbf{R} \in \mathcal{R}^n, \forall x, y \in X : [x_{i_0} P_{i_0} y_{i_0} \text{ and } \nexists i \in N \text{ with } x_i \; I_i \; y_i] \Rightarrow x \bar{P}(\mathbf{R}) y.$$

Lemma 3 *If a SOF $\bar{R}$ satisfies Weak Pareto and Weak IIA, then it is quasi-dictatorial.*

Proof. Let $\bar{R}$ be a SOF that satisfies Weak Pareto and Weak IIA. By an adaptation of a standard proof of Arrow's theorem (for instance, Sen [18]), we can show that for every free triple $Y \subset X$, there exists a quasi-dictator over $(Y, \mathcal{R}^n)$. Then, a direct application of Bordes and Le Breton [3] establishes quasi-dictatorship of $\bar{R}$. ■

It is interesting that in our economic environments, quasi-dictatorship is equivalent to dictatorship as the next lemma shows.

Lemma 4 *If a SOF $\bar{R}$ is quasi-dictatorial, then it is dictatorial.*

Proof. Let $\bar{R}$ be a quasi-dictatorial SOF. Let $x, y \in X$ and $\mathbf{R} \in \mathcal{R}^n$ be such that $x_{i_0}\ P_{i_0}\ y_{i_0}$. By continuity and strict monotonicity of preferences, there exists $x' \in X$ such that $x_{i_0}\ P_{i_0}\ x'_{i_0}\ P_{i_0}\ y_{i_0}$ and for all $i \in N$, either $x_i\ P_i\ x'_i\ P_i\ y_i$ or $y_i\ R_i\ x_i\ P_i\ x'_i$. Since $\bar{R}$ is quasi-dictatorial, it follows that $x\ \bar{P}(\mathbf{R})\ x'$ and $x'\ \bar{P}(\mathbf{R})\ y$. By transitivity, $x\ \bar{P}(\mathbf{R})\ y$. ■

Given these lemmas, the proof of Proposition 4 is straightforward.

Proof of Proposition 4: Let $\bar{R}$ be a SOF that satisfies Weak Pareto and IIA-ISFA. By Lemma 2, $\bar{R}$ satisfies WIIA. Then, by Lemmas 3 and 4, $\bar{R}$ is dictatorial. ■

The proof of this proposition can be immediately adapted to extend the result to IIA-ISλEFA.

Combining the first and the second extensions of informational basis amounts to taking indifference hypersurfaces in some finitely sized ε-neighborhoods of allocations as well as in some radial expansion of the "Edgeworth box". By this extension, we allow the social ranking of any two allocations to depend on the shapes of very large portions of indifference hypersurfaces around the allocations, and hence the independence condition becomes very weak. However, incompatibility with anonymity persists no matter how large the value of ε is.

Proposition 5 *Let $\varepsilon > 0$ and $\lambda \geq 1$ be given. There exists a SOF that satisfies Weak Pareto, IIA-IS[εNλEFA], and is not dictatorial. However, there exists no SOF that satisfies Weak Pareto, IIA-IS[εNλEFA] and Anonymity.*

The possibility part is directly implied by Proposition 3 because IIA-ISεN implies IIA-IS[εNλEFA]. The proof of the impossibility part is in the appendix.

Going to the third direction of extending informational basis, and following Pazner and Schmeidler's [16] contribution, we can derive the next result, which shows that not much information is needed to have an anonymous SOF if only we are prepared to accept an externally specified reference bundle, although it must be substantially non-local information.

Proposition 6 *There exists a SOF that satisfies Weak Pareto, IIA-ISPω_0 and Anonymity.*

Proof. By continuity and strict monotonicity of preferences, the following utility functions

$$u_i(x_i) = \min\{\alpha \in \mathbb{R}_+ | \alpha\omega_0 R_i x_i\}$$

are well-defined and represent preferences R_i. Let $\bar{R}$ be defined by:

$$x\bar{R}(\mathbf{R})y \Leftrightarrow \min\{u_i(x_i)|i \in N\} \geq \min\{u_i(y_i)|i \in N\}.$$

This SOF clearly satisfies Weak Pareto and Anonymity. It also satisfies IIA-ISPω_0 because whenever $I(x_i, R_i) \cap \Lambda_{\omega_0} = I(x_i, R_i') \cap \Lambda_{\omega_0}$, we have

$$\min\{\alpha \in \mathbb{R}_+ | \alpha\omega_0 R_i x_i\} = \min\{\alpha \in \mathbb{R}_+ | \alpha\omega_0 R_i' x_i\}.$$

■

Since IIA-ISPω_0 implies IIA-WIS, we also have the following corollary.

Corollary 1 *There exists a SOF that satisfies Weak Pareto, IIA-WIS and Anonymity.*

Notice that we could have the Strong Pareto property[2] as well by relying on the leximin criterion rather than the maximin for the SOF defined in the above proof. There are also many examples of SOFs satisfying Weak Pareto, IIA-WIS and Anonymity. Thus, in addition to these three axioms, we may add other requirements embodying various equity principles.[3]

4 Conclusion

The construction of a non-dictatorial Arrovian social ordering function, in a framework with purely ordinal, interpersonally non-comparable preferences, requires information about the shape of indifference curves that goes well beyond infinitesimally local data such as marginal rates of substitution or data

[2]**Strong Pareto**: $\forall x, y \in \mathbb{R}_+^{n\ell}, \forall \mathbf{R} \in \mathcal{R}^n$ if $\forall i \in N,\ x_i R_i y_i$, then $x\bar{R}(\mathbf{R})y$ and if, in addition, $\exists i \in N,\ x_i P_i y_i$, then $x\bar{P}(\mathbf{R})y$.

[3]Notice that Strong Pareto and Anonymity already entail a version of the Suppes grading principle: for all $\mathbf{R} \in \mathcal{R}^n$, all x, y, if there are i, j such that $R_i = R_j$, $x_i P_i y_j$ and $x_j P_i y_i$, and for $h \neq i, j$, $x_h = y_h$, then $x\bar{P}(\mathbf{R})y$. Notice also that it is easy to construct SOFs satisfying Strong Pareto, IIA-WIS (or IIA-ISPω_0), Anonymity and the following version of the Hammond equity axiom (Hammond [10]): for all $\mathbf{R} \in \mathcal{R}^n$, and all $x, y \in \mathbb{R}_+^{n\ell}$, if there are i, j such that $R_i = R_j$, $y_i P_i x_i P_i x_j P_i y_j$, and for all $h \neq i, j$, $x_h = y_h$, then $x\bar{P}(\mathbf{R})y$.

"within the Edgeworth box". On the basis of information in some finitely sized neighborhoods, one can construct a non-dictatorial social ordering function, but still cannot have an anonymous one. Only substantially non-local information about indifference curves enables one to construct a Paretian and anonymous social ordering function. These are the main messages of this paper, in which we proved two extensions of Arrow's impossibility theorem, and several possibility results. We hope that our paper, more broadly, contributes to clarifying the informational foundations in the theory of social choice.

There are limits to our work which may be noticed here, and call for further research. First, we study a particular economic model, and it would be worth analyzing the same issues in other models such as the standard abstract model of social choice or other economic models, in particular models with public goods (the case of consumption externalities in our model could also be subsumed under the case of public goods). Second, the information about indifference curves is a complex set of object, and our analysis is far from being exhaustive on the pieces of data which can be extracted from this set. We have focussed on what seemed to us the most natural parts of indifference curves to which one may want to refer in social evaluation of allocations, namely, the marginal rates of substitution, the Edgeworth box (bundles which are achievable by redistributing the considered allocations), and reference rays. But there may be other ways of considering indifference curves. For instance, it would be nice to have a measure of the degree to which a given piece of information is local, and the connection between this work and topological social choice (e.g. Chichilnisky [6]) might be worth exploring. Third, there may be other kinds of interesting additional information. For instance, Roberts [17] considered introducing information about utilities and about non-local preferences at the same time, and was able to characterize the Nash social welfare function on this basis. There certainly are many avenues of research along these lines. The purpose of this paper would be well-served if it could open the gate towards these enticing avenues.

References

1. K. J. Arrow, A difficulty in the concept of social welfare, *J. Political Economy* **58** (1950), 328-346.

2. K. J. Arrow, "Social Choice and Individual Values," Wiley, New York, 1951. Second ed., 1963.

3. G. Bordes and M. Le Breton, Arrovian theorems with private alternatives domains and selfish individuals, *J. Econ. Theory* **47** (1989), 257-281.

4. W. Bossert, M. Fleurbaey and D. Van de gaer, Responsibility, talent, and compensation: a second-best analysis, *Review of Economic Design* **4** (1999), 35-55.

5. D. E. Campbell and J. S. Kelly, Information and preference aggregation, *Social Choice and Welfare* **17** (2000), 3-24.

6. G. Chichilnisky, A unified perspective on resource allocation: limited arbitrage is necessary and sufficient for the existence of a competitive equilibrium, the core and social choice, *in* "Social Choice Re-examined" (K. J. Arrow, A. K. Sen and K. Suzumura, Eds.), Vol. 1, pp. 121-174, Macmillan, London, and St Martin's Press, New York, 1997.

7. C. d'Aspremont and L. Gevers, Equity and the informational basis of collective choice, *Review of Economic Studies* **44** (1977), 199-209.

8. M. Fleurbaey and F. Maniquet, Utilitarianism versus fairness in welfare economics, 1996, forthcoming *in* "Justice, Political Liberalism and Utilitarianism: Themes from Harsanyi and Rawls" (M. Salles and J. A. Weymark, Eds), Cambridge University Press, Cambridge.

9. M. Fleurbaey and F. Maniquet, Fair social orderings with unequal production skills, mimeo, U. of Cergy, 2000.

10. P. Hammond, Equity, Arrow's conditions, and Rawls' difference principle, *Econometrica* **44** (1976), 793-804.

11. B. Hansson, The independence condition in the theory of social choice, *Theory and Decision* **4** (1973), 25-49.

12. K.-I. Inada, On the economic welfare function, *Econometrica* **32** (1964), 316-338.

13. E. Kalai, E. Muller and M. A. Satterthwaite, Social welfare functions when preferences are convex, strictly monotonic, and continuous, *Public Choice* **34** (1979), 87-97.

14. F. Maniquet, "On Equity and Implementation in Economic Environments," Ph.D. Thesis, University of Namur, 1994.

15. E. Pazner, Equity, nonfeasible alternatives and social choice: a reconsideration of the concept of social welfare, *in* "Aggregation and Revelation of Preferences" (J. J. Laffont, Ed.), North-Holland, Amsterdam, 1979.

16. E. Pazner and D. Schmeidler, Egalitarian-equivalent allocations: a new concept of economic equity, *Quart. J. Econ.* **92** (1978), 671-687.

17. K. Roberts, Interpersonal comparability and social choice theory, *Rev. Econ. Studies* **47** (1980), 421-439.

18. A. K. Sen, "Collective choice and social welfare," Holden-Day, San-Francisco, 1970, and republished North-Holland Amsterdam, 1979.

19. K. Suzumura, On Pareto-efficiency and the no-envy concept of equity, *J. Econ. Theory* **25** (1981), 367-379.

20. K. Suzumura, On the possibility of 'fair' collective choice rule, *Int. Econ. Rev.* **22** (1981), 351-364.

21. K. Tadenuma, Efficiency first or equity first? Two principles and rationality of social choice, *J. Econ. Theory* **104** (2002), 462-472.

Appendix

A.1 Proof of Proposition 2

The proof of Proposition 2 relies on the following lemmas.

Let $Y \subset X$ be a given finite subset of X. Let $i \in N$ be given. Define $Y_i = \{y_i \in \mathbb{R}^\ell_+ |\ \exists y_{-i} \in \mathbb{R}^{(n-1)\ell}_+, (y_i, y_{-i}) \in Y\}$. Let $\mathcal{Q}$ denote the set of convex cones in $\mathbb{R}^\ell_{++}$. For each $y_i \in Y_i$, let $Q(y_i) \in \mathcal{Q}$ be given. We say that the set Y_i *satisfies the supporting condition with respect to* $\{Q(y_i)|\ y_i \in Y_i\}$ if for all $y_i \in Y_i$, all $q \in Q(y_i)$, and all $y'_i \in Y_i$ with $y'_i \neq y_i$, $q \cdot y_i < q \cdot y'_i$. Define

$$\mathcal{R}(Y_i, \{Q(y_i)|\ y_i \in Y_i\}) = \{R_i \in \mathcal{R}|\ \forall y_i \in Y_i, C(y_i, R_i) = Q(y_i)\}.$$

The set of all preorderings on Y_i is denoted by $\mathcal{O}(Y_i)$. For any $R_i \in \mathcal{R}$, $R_i|_{Y_i}$ denotes the restriction of R_i on Y_i[4]. For any $\mathcal{R}' \subset \mathcal{R}$, let $\mathcal{R}'|_{Y_i} = \{R_i|_{Y_i} \mid R_i \in \mathcal{R}'\}$. For any $x_i \in X$ and any $R_i \in \mathcal{R}$, let $U(x_i, R_i) = \{\ x'_i \in X \mid x'_i\ R_i\ x_i\ \}$ denote the (closed) upper contour set of x_i for R_i.

Lemma 5 *If a finite set* $Y_i \subset \mathbb{R}^\ell_+$ *satisfies the supporting condition with respect to* $\{Q(y_i)|\ y_i \in Y_i\}$*, then* $\mathcal{R}(Y_i, \{Q(y_i)|y_i \in Y_i\})|_{Y_i} = \mathcal{O}(Y_i)$.

Proof. We have only to show that $\mathcal{O}(Y_i) \subseteq \mathcal{R}(Y_i, \{Q(y_i)|y_i \in Y_i\})|_{Y_i}$. Let R' $\in \mathcal{O}(Y_i)$ be any preordering on Y_i. Construct a preordering $R_i \in \mathcal{R}$ so that the upper contour set of each $y_i \in Y_i$ is defined as follows. Let $x_i \in Y_i$ be such that for all $y_i \in Y_i$, $y_i\ R'_i\ x_i$. Define $Y^1_i = \{y_i \in Y_i \mid y_i\ I'_i\ x_i\}$. For each $a \in \mathbb{R}^\ell_+$ and each $q \in \mathbb{R}^\ell_{++}$, define $H(a, q) = \{b \in \mathbb{R}^\ell_+ \mid q \cdot b \geq q \cdot a\}$. Let

$$U(x_i, R_i) = \bigcap_{y_i \in Y^1_i} \left[\bigcap_{q \in Q(y_i)} H(y_i, q) \right]$$

Let $I(x_i, R_i)$ be the boundary of $U(x_i, R_i)$. Clearly, for all $y_i \in Y^1_i$, $C(y_i, R_i)$ $= Q(y_i)$. We also have that for all $y_i \in Y_i \backslash Y^1_i$, and for all $x'_i \in I(x_i, R_i)$, $y_i\ P_i\ x'_i$. Given $\delta > 0$, let $(1+\delta)U(x_i, R_i) = \{x'_i \in \mathbb{R}^\ell_+ \mid \exists a_i \in U(x_i, R_i),$ $x'_i = (1+\delta)a_i\}$, and let $(1+\delta)I(x_i, R_i)$ be the boundary of $(1+\delta)U(x_i, R_i)$. For sufficiently small δ, we have that for all $y_i \in Y_i \backslash Y^1_i$, and for all $x'_i \in$

[4]Namely, $R_i|_{Y_i}$ is the preordering on Y_i such that for all $x_i, y_i \in Y_i$, $x_i\ R_i|_{Y_i}\ y_i \iff x_i R_i y_i$.

$(1+\delta)I(x_i, R_i)$, $y_i P_i x_i'$. Let $z_i \in Y_i \backslash Y_i^1$ be such that for all $y_i \in Y_i \backslash Y_i^1$, $y_i\ R_i'\ z_i$. Define $Y_i^2 = \{y_i \in Y_i \backslash Y_i^1 \mid y_i\ I_i'\ z_i\}$. Let

$$U(z_i, R_i) = (1+\delta)U(x_i, R_i) \bigcap \left(\bigcap_{y_i \in Y_i^2} \left[\bigcap_{q \in Q(y_i)} H(y_i, q) \right] \right)$$

Let $I(z_i, R_i)$ be the boundary of $U(z_i, R_i)$. By definition, for all $y_i \in Y_i^2$, $C(y_i, R_i) = Q(y_i)$. We have that for all $y_i \in Y_i \backslash \left(Y_i^1 \cup Y_i^2\right)$, and for all $x_i' \in I(z_i, R_i)$, $y_i\ P_i\ x_i'$. Similarly we can construct the upper contour set of each $y_i \in Y_i \backslash \left(Y_i^1 \cup Y_i^2\right)$. By its construction, $R_i \in \mathcal{R}(Y_i, \{Q(y_i) | y_i \in Y_i\})$ and $R_i|_{Y_i} = R'$. Thus, $R' \in \mathcal{R}(Y_i, \{Q(y_i) | y_i \in Y_i\})|_{Y_i}$. ■

Let $\bar{R}$ be a social ordering function. Let $Y \subseteq X$ and $\mathcal{R}' \subseteq \mathcal{R}^n$ be given. We say that agent $i_0 \in N$ is a *local dictator* for $\bar{R}$ over $(Y, \mathcal{R}')$ if for all $x, y \in Y$, and all $\mathbf{R} \in \mathcal{R}'$, $x_{i_0} P_{i_0} y_{i_0}$ implies $x \bar{P}(\mathbf{R}) y$.

Lemma 6 *Let $\bar{R}$ be a social ordering function satisfying Weak Pareto and IIA-MRS. Let $Y \subset X$ be a finite subset of X such that $|Y| \geq 3$.*[5] *Suppose that for all $i \in N$, Y_i satisfies the supporting condition with respect to $\{Q(y_i) |\ y_i \in Y_i\}$. Then, there exists a local dictator $i_0 \in N$ for $\bar{R}$ over $(Y, \prod_{i \in N} \mathcal{R}(Y_i, \{Q(y_i) |\ y_i \in Y_i\}))$.*

Proof. For all $\mathbf{R}, \mathbf{R}' \in \prod_{i \in N} \mathcal{R}(Y_i, \{Q(y_i) |\ y_i \in Y_i\})$, all $y \in Y$, and all $i \in N$, $C(y_i, R_i) = C(y_i, R_i')$. Since $\bar{R}$ satisfies IIA-MRS, we have that for all $x, y \in Y$, and all $\mathbf{R}, \mathbf{R}' \in \prod_{i \in N} \mathcal{R}(Y_i, \{Q(y_i) |\ y_i \in Y_i\})$, if for all $i \in N$, R_i and R_i' agree on $\{x_i, y_i\}$, then $\bar{R}(\mathbf{R})$ and $\bar{R}(\mathbf{R}')$ agree on $\{x, y\}$. By Lemma 5, for all $i \in N, \mathcal{R}(Y_i, \{Q(y_i) | y_i \in Y_i\})|_{Y_i} = \mathcal{O}(Y_i)$. Hence, by Arrow's Theorem, there exists a local dictator for $\bar{R}$ over $(Y, \prod_{i \in N} \mathcal{R}(Y_i, \{Q(y_i) |\ y_i \in Y_i\}))$. ■

We say that a subset Y of X is *free for agent i* if $\mathcal{R}|_{Y_i} = \mathcal{O}(Y_i)$. It is *free* if it is free for all $i \in N$. If Y contains two elements, it is a *free pair*. If Y contains three elements, it is a *free triple*. Note that a set $\{x, y\}$ is a free pair for $i \in N$ if and only if for some $k, k' \in \{1, \cdots, \ell\}$, $x_{ik} > y_{ik}$ and $y_{ik'} > x_{ik'}$. Given two consumption bundles $x_i, y_i \in \mathbb{R}_+^\ell$, define $x_i \wedge y_i \in \mathbb{R}_+^\ell$ as $(x_i \wedge y_i)_k = \min\{x_{ik}, y_{ik}\}$ for all $k \in \{1, \cdots, \ell\}$.

Lemma 7 *Let $\bar{R}$ be a social ordering function satisfying Weak Pareto and IIA-MRS. If $\{x, y\} \subset X$ is a free pair, then there exists a local dictator for $\bar{R}$ over $(\{x, y\}, \mathcal{R}^n)$.*

[5] Given a set A, $|A|$ denotes the cardinality of A.

Proof. Let $\bar{R}$ be a social ordering function satisfying Weak Pareto and IIA-MRS. Let $\{x, y\} \subset X$ be a free pair. Let

$$\begin{aligned} K_1 &= \{k \in \{1, \cdots, \ell\} \mid x_{ik} > y_{ik}\} \\ K_2 &= \{k \in \{1, \cdots, \ell\} \mid x_{ik} < y_{ik}\} \end{aligned}$$

Since $\{x, y\}$ is a free pair, $K_1, K_2 \neq \emptyset$.

Step 1 : For each $i \in N$, we define two consumption bundles $z_i, w_i \in X$ as follows:

$$z_i = x_i \wedge y_i + \frac{1}{2}\left[\frac{2}{3}(x_i - x_i \wedge y_i) + \frac{1}{3}(y_i - x_i \wedge y_i)\right] \quad (1)$$

$$w_i = x_i \wedge y_i + \frac{1}{2}\left[\frac{1}{3}(x_i - x_i \wedge y_i) + \frac{2}{3}(y_i - x_i \wedge y_i)\right] \quad (2)$$

Figure 1 illustrates the bundles $x_i, y_i, x_i \wedge y_i, z_i, w_i$, and also b_i, v_i, t_i, which are defined in the next step. Let $q \in \mathbb{R}^{\ell}_{++}$. Then, $q \cdot y_i < q \cdot w_i$ if and only if

$$\frac{2}{3}\sum_{k \in K_2} q_k(y_{ik} - x_{ik}) < \frac{1}{6}\sum_{k \in K_1} q_k(x_{ik} - y_{ik}) \quad (3)$$

Since $K_1 \neq \emptyset$, the right-hand-side of (3) can be arbitrarily large as $(q_k)_{k \in K_1}$ become large, $(q_k)_{k \in K_2}$ being constant. Hence, there exists a price vector $q(y_i) \in \mathbb{R}^{\ell}_{++}$ that satisfies inequality (3). With some calculations, it can be shown that $q(y_i) \cdot y_i < q(y_i) \cdot z_i$ and $q(y_i) \cdot y_i < q(y_i) \cdot x_i$.

Similarly, for each $a \in \{x_i, z_i, w_i\}$, we can find a price vector $q(a) \in \mathbb{R}^{\ell}_{++}$ such that for all $a' \in \{x_i, z_i, w_i, y_i\}$ with $a' \neq a$, $q(a) \cdot a < q(a) \cdot a'$. Hence, the set $Y_i^0 = \{x_i, z_i, w_i, y_i\}$ satisfies the supporting condition with respect to $\{q(x_i), q(z_i), q(w_i), q(y_i)\}$.[6]

Let $z = (z_i)_{i \in N}$ and $w = (w_i)_{i \in N}$. Let $Y^0 = \{x, z, w, y\}$. By Lemma 6, there exists a local dictator $i_0 \in N$ for $\bar{R}$ over $(Y^0, \prod_{i \in N} \mathcal{R}(Y_i^0, \{q(x_i), q(z_i), q(w_i), q(y_i)\})$.

Step 2: We will show that agent i_0 is a local dictator for $\bar{R}$ over $(\{x, y\}, \mathcal{R}^n)$.

Suppose, on the contrary, that there exists a preference profile $\mathbf{R}^0 \in \mathcal{R}^n$ such that (i) $x_{i_0} P^0_{i_0} y_{i_0}$ and $y \bar{R}(\mathbf{R}^0) x$ or (ii) $y_{i_0} P^0_{i_0} x_{i_0}$ and $x \bar{R}(\mathbf{R}^0) y$. Without loss of generality, suppose that (i) holds. Let $Y^1 = \{z, w, y\}$. Since agent i_0 is the local dictator for $\bar{R}$ over $(Y^0, \prod_{i \in N} \mathcal{R}(Y_i^0, \{q(x_i), q(z_i), q(w_i), q(y_i)\})$, he

[6] With a slight abuse of notation, we are writing $q(\cdot)$ for $Q(\cdot) = \{\alpha q(\cdot) \mid \alpha > 0\}$.

is also the local dictator for $\bar{R}$ over $(Y^1, \prod_{i\in N} \mathcal{R}(Y_i^1, \{q(z_i), q(w_i), q(y_i)\})$. (Otherwise, by Lemma 6, there exists a local dictator $j \neq i_0$ for $\bar{R}$ over $(Y^1, \prod_{i\in N} \mathcal{R}(Y_i^1, \{q(z_i), q(w_i), q(y_i)\})$, and we can construct a preference profile $\mathbf{R} \in \prod_{i\in N} \mathcal{R}(Y_i^0, \{q(x_i), q(z_i), q(w_i), q(y_i)\}) \subset \prod_{i\in N} \mathcal{R}(Y_i^1, \{q(z_i), q(w_i), q(y_i)\})$ such that $z_{i_0} P_{i_0} w_{i_0}$ and $w_j P_j z_j$. Hence we must have $z\bar{P}(\mathbf{R})w$ and $w\bar{P}(\mathbf{R})z$, which is a contradiction.)

We define two allocations $v, t \in X$ in the following steps. Let $i \in N$. First, define $b_i \in \mathbb{R}_+^\ell$ as follows: If for all $q \in C(x_i, R_i^0)$, $q \cdot (y_i - x_i) \geq 0$, then let $b_i = y_i$. If for some $q \in C(x_i, R_i^0)$, $q \cdot (y_i - x_i) < 0$, then let $\theta > 0$ be a positive number such that for all $q \in C(x_i, R_i^0)$, $q \cdot [y_i + \theta(y_i - x_i \wedge y_i) - x_i] > 0$. Since $q \in \mathbb{R}_{++}^\ell$ by strict monotonicity of preferences, and $y_i - x_i \wedge y_i > 0$, such a number θ exists. Then, define $b_i = y_i + \theta(y_i - x_i \wedge y_i)$. By definition, $b_i > y_i$, and for all $q \in C(x_i, R_i^0)$, $q \cdot (b_i - x_i) > 0$. Define

$$v_i = b_i + 2(b_i - x_i \wedge y_i)$$

Then, $v_i > b_i > y_i$, and for all $q \in C(x_i, R_i^0)$, $q \cdot (v_i - x_i) > 0$.

Next, define

$$t_i = x_i \wedge y_i + \frac{1}{2}\left[\frac{2}{3}(v_i - x_i \wedge y_i) + \frac{1}{3}(w_i - x_i \wedge y_i)\right]$$

Then,

$$t_i = b_i + \frac{1}{6}(w_i - x_i \wedge y_i) > b_i$$

and for all $q \in C(x_i, R_i^0)$, $q \cdot x_i < q \cdot t_i$.

As in Step 1, we can find price vectors $q(v_i), q(t_i) \in \mathbb{R}_{++}^\ell$ such that $q(v_i) \cdot v_i < q(v_i) \cdot a$ for all $a \in \{x_i, z_i, w_i, t_i\}$, and $q(t_i) \cdot t_i < q(t_i) \cdot a$ for all $a \in \{x_i, z_i, w_i, v_i\}$.

On the other hand, because $v_i > y_i$ and $t_i > y_i$, we have that $q(z_i) \cdot z_i < q(z_i) \cdot a$ for all $a \in \{t_i, v_i\}$, and $q(w_i) \cdot w_i < q(w_i) \cdot a$ for all $a \in \{t_i, v_i\}$.

So far we have shown that

(i) the set $Y_i^1 = \{x_i, t_i, v_i\}$ satisfies the supporting condition with respect to $\{C(x_i, R_i^0), q(t_i), q(v_i)\}$.

(ii) the set $Y_i^2 = \{z_i, w_i, t_i, v_i\}$ satisfies the supporting condition with respect to $\{q(z_i), q(w_i), q(t_i), q(v_i)\}$.

Let $v = (v_i)_{i\in N}$ and $t = (t_i)_{i\in N}$. Let $Y^1 = \{x, t, v\}$ and $Y^2 = \{z, w, t, v\}$. By Lemma 6, there exist a local dictator $i_1 \in N$ for $\bar{R}$ over $(Y^1, \prod_{i\in N} \mathcal{R}(Y_i^1, \{C(x_i, R_i^0), q(t_i), q(v_i)\})$, and a local dictator $i_2 \in N$ for $\bar{R}$

over $(Y^2, \prod_{i \in N} \mathcal{R}(Y_i^2, \{q(z_i), q(w_i), q(t_i), q(v_i)\})$. Recall that agent $i_0 \in N$ is the local dictator for $\bar{R}$ over $(Y^0, \prod_{i \in N} \mathcal{R}(Y_i^0, \{q(z_i), q(w_i), q(y_i)\})$. Let $\mathbf{R}^1 \in \mathcal{R}^n$ be a preference profile such that for all $i \in N$, $C(x_i, R_i^1) = C(x_i, R_i^0)$, and for all $a_i \in \{t_i, v_i, w_i, y_i, z_i\}$, $C(a_i, R_i^1) = \{q(a_i)\}$, and such that

$$\begin{aligned} & x_{i_0} P_{i_0}^1 z_{i_0} P_{i_0}^1 w_{i_0} P_{i_0}^1 t_{i_0} P_{i_0}^1 v_{i_0} P_{i_0}^1 y_{i_0} \\ \text{and for all } i \in N \text{ with } i \neq i_0,\ & x_i P_i^1 v_i P_i^1 t_i P_i^1 w_i P_i^1 z_i P_i^1 y_i \end{aligned}$$

Since $\mathbf{R}^1 \in \prod_{i \in N} \mathcal{R}(Y_i^0, \{q(z_i), q(w_i), q(y_i)\}$, and agent i_0 is the local dictator for $\bar{R}$ over $(Y^0, \prod_{i \in N} \mathcal{R}(Y_i^0, \{q(z_i), q(w_i), q(y_i)\})$, we have $z\bar{P}(\mathbf{R}^1)w$. Because $\mathbf{R}^1 \in \prod_{i \in N} \mathcal{R}(Y_i^2, \{q(z_i), q(w_i), q(t_i), q(v_i)\}$, this implies that $i_0 = i_2$. Hence, we have $t\bar{P}(\mathbf{R}^1)v$. Since $\mathbf{R}^1 \in \prod_{i \in N} \mathcal{R}(Y_i^1, \{C(x_i, R_i^0), q(t_i), q(v_i)\}$, it follows that $i_0 = i_1$.

Let $\mathbf{R}^2 \in \mathcal{R}^n$ be a preference profile such that $x_{i_0} P_{i_0}^2 v_{i_0}$ and for all $i \in N$, $R_i^2|_{\{x_i, y_i\}} = R_i^0|_{\{x_i, y_i\}}$, and $C(x_i, R_i^2) = C(x_i, R_i^0), C(t_i, R_i^2) = \{q(t_i)\}, C(v_i, R_i^2) = \{q(v_i)\}$ and $C(y_i, R_i^2) = C(y_i, R_i^0)$. Since agent $i_0 \in N$ is the local dictator for $\bar{R}$ over $(Y^1, \prod_{i \in N} \mathcal{R}(Y_i^1, \{C(x_i, R_i^0), q(t_i), q(v_i)\})$, we have that $x\bar{P}(\mathbf{R}^2)v$. Recall that for all $i \in N$, $v_i > y_i$. Hence, by strict monotonicity of preferences, $v_i P_i^2 y_i$ for all $i \in N$. Because the social ordering function $\bar{R}$ satisfies Weak Pareto, we have $v\bar{P}(\mathbf{R}^2)y$. By transitivity of $\bar{R}$, $x\bar{P}(\mathbf{R}^2)y$. However, since $\bar{R}$ satisfies IIA-MRS, and $y\bar{R}(\mathbf{R}^0)x$, we must have $y\bar{R}(\mathbf{R}^2)x$. This is a contradiction. ■

Lemma 8 *Let $\bar{R}$ be a social ordering function satisfying Weak Pareto and IIA-MRS. If $\{x, y, z\} \subset X$ is a free triple, then there exists a local dictator for $\bar{R}$ over $(\{x, y, z\}, \mathcal{R}^n)$.*

Proof. By Lemma 7, there exist a local dictator i_0 over $(\{x, y\}, \mathcal{R}^n)$, a local dictator i_1 over $(\{y, z\}, \mathcal{R}^n)$, and a local dictator i_2 over $(\{x, z\}, \mathcal{R}^n)$. Suppose that $i_0 \neq i_1$. Let $\mathbf{R} \in \mathcal{R}^n$ be a preference profile such that $x_{i_0} P_{i_0} y_{i_0}$, $y_{i_1} P_{i_1} z_{i_1}$, and $z_{i_2} P_{i_2} x_{i_2}$. Then, we have $x\bar{P}(\mathbf{R})y\bar{P}(\mathbf{R})z\bar{P}(\mathbf{R})x$, which contradicts the transitivity of $\bar{R}(\mathbf{R})$. Hence, we must have $i_0 = i_1$. By the same argument, we have $i_0 = i_1 = i_2$. ■

Proof of Proposition 2: Let $\bar{R}$ be a social ordering function satisfying Weak Pareto and IIA-MRS. By Lemma 7, for every free pair $\{x, y\} \subset X$, there exists a local dictator over $(\{x, y\}, \mathcal{R}^n)$. By Lemma 8 and Bordes and Le Breton [3, Theorem 2], these dictators must be the same individual. Denote the individual by i_0. It remains to show that for any pair $\{x, y\}$ that is not free, i_0 is the local dictator over $(\{x, y\}, \mathcal{R}^n)$. Suppose, on the contrary, that there exist $\{x, y\} \subset X$ and $\mathbf{R} \in \mathcal{R}^n$ such that $\{x, y\}$ is not a free pair, and $x_{i_0} P_{i_0} y_{i_0}$ but $y \bar{R}(\mathbf{R}) x$. Define $z_{i_0} \in \mathbb{R}^\ell_+$ as follows.

Case 1: $\{x, y\}$ is a free pair for i_0.

For all $\lambda \in]0, 1[$, $\{\lambda x + (1-\lambda)y, x\}$ and $\{\lambda x + (1-\lambda)y, y\}$ are free pairs for i_0. By continuity, there exists λ^* such that $x_{i_0} P_{i_0} [\lambda^* x_{i_0} + (1-\lambda^*) y_{i_0}] P_{i_0} y_{i_0}$. Then, let $z_{i_0} = \lambda^* x_{i_0} + (1-\lambda^*) y_{i_0}$.

Case 2: $\{x, y\}$ is not a free pair for i_0.

Then, for all $k \in \{1, \cdots, \ell\}$, $x_{i_0 k} \geq y_{i_0 k}$ with at least one strict inequality. Note that $y \neq 0$.

Case 2-1: There exists k' such that for all $k \in \{1, \cdots, \ell\}$ with $k \neq k'$, $x_{i_0 k} = y_{i_0 k}$ and $y_{i_0 k'} > 0$.

Then, $x_{i_0 k'} > y_{i_0 k'} > 0$. Given $\varepsilon > 0$, define $w_{i_0} \in \mathbb{R}^\ell_+$ as $w_{i_0 k'} = y_{i_0 k'}$ and for all $k \neq k', w_{i_0 k} = y_{i_0 k} + \varepsilon$. For sufficiently small ε, we have $x_{i_0} P_{i_0} w_{i_0} P_{i_0} y_{i_0}$. Given $\delta > 0$, define $t_{i_0} \in \mathbb{R}^\ell_+$ as $t_{i_0 k'} = w_{i_0 k'} - \delta$ and for all $k \neq k', t_{i_0 k} = w_{i_0 k}$. For sufficiently small δ, we have $x_{i_0} P_{i_0} t_{i_0} P_{i_0} y_{i_0}$. Moreover, $\{t, x\}$ and $\{t, y\}$ are free pairs for i_0. Then, let $z_{i_0} = t_{i_0}$.

Case 2-2: There exists k' such that for all $k \in \{1, \cdots, \ell\}$ with $k \neq k'$, $x_{i_0 k} = y_{i_0 k}$ and $y_{i_0 k'} = 0$.

Then, for all $k \in \{1, \cdots, \ell\}$ with $k \neq k'$, $x_{i_0 k} = y_{i_0 k} > 0$. Let $k'' \neq k'$. Given $\varepsilon > 0$, define $w_{i_0} \in \mathbb{R}^\ell_+$ as $w_{i_0 k''} = x_{i_0 k''} - \varepsilon$ and for all $k \neq k'', w_{i_0 k} = x_{i_0 k}$. For sufficiently small ε, we have $x_{i_0} P_{i_0} w_{i_0} P_{i_0} y_{i_0}$. Given $\delta > 0$, define $t_{i_0} \in \mathbb{R}^\ell_+$ as $t_{i_0 k'} = w_{i_0 k'} + \delta$ and for all $k \neq k', t_{i_0 k} = w_{i_0 k}$. For sufficiently small δ, we have $x_{i_0} P_{i_0} t_{i_0} P_{i_0} y_{i_0}$. Moreover, $\{t, x\}$ and $\{t, y\}$ are free pairs for i_0. Then, let $z_{i_0} = t_{i_0}$.

Case 2-3: There exist $k', k'' \in \{1, \cdots, \ell\}$ with $k' \neq k''$, $x_{i_0 k'} > y_{i_0 k'}$ and $x_{i_0 k''} > y_{i_0 k''}$.

Let k^* be such that $y_{i_0 k^*} > 0$. Given $\varepsilon > 0$, define $w_{i_0} \in \mathbb{R}^\ell_+$ as $w_{i_0 k^*} = y_{i_0 k^*} - \varepsilon$ and for all $k \neq k^*, w_{i_0 k} = x_{i_0 k}$. For sufficiently small ε, we have $x_{i_0} P_{i_0} w_{i_0} P_{i_0} y_{i_0}$. Let $k^{**} \neq k^*$. Given $\delta > 0$, define $t_{i_0} \in \mathbb{R}^\ell_+$ as $t_{i_0 k^{**}} = w_{i_0 k^{**}} + \delta$ and for all $k \neq k^{**}, t_{i_0 k} = w_{i_0 k}$. For sufficiently small δ, we have $x_{i_0} P_{i_0} t_{i_0} P_{i_0} y_{i_0}$. Moreover, $\{t, x\}$ and $\{t, y\}$ are free pairs for i_0. Then, let $z_{i_0} = t_{i_0}$.

Next, for each $i \neq i_0$, let $z_i \in \mathbb{R}^\ell_+$ be such that $\{z, x\}$ and $\{z, y\}$ are free pairs for i. By the same construction as above, we can find such $z_i \in \mathbb{R}^\ell_+$ for each i. Let $z = (z_i)_{i \in N} \in \mathbb{R}^{n\ell}_+$. Since i_0 is the dictator over all free pairs, we have that $x \bar{P}(\mathbf{R}) z$ and $z \bar{P}(\mathbf{R}) y$. By transitivity of $\bar{R}$, we have $x \bar{P}(\mathbf{R}) y$, which contradicts the supposition that $y \bar{R}(\mathbf{R}) x$.■

A.2 Proof of Lemma 2

To prove Lemma 2, we need an auxiliary lemma. Define

$$\begin{aligned} X_1 &= \{x_i \in \mathbb{R}^\ell_+ \setminus \{0\} \mid \forall k \geq 2, x_{ik} = 0\} \\ X_2 &= \{x_i \in \mathbb{R}^\ell_+ \setminus \{0\} \mid \forall k \neq 2, x_{ik} = 0\}. \end{aligned}$$

Lemma 9 *For all $R_i \in \mathcal{R}$, and all $x, y \in X$, there exists $R_i^* \in \mathcal{R}$ such that*

$$\begin{aligned} I(x_i, R_i) \cap \Omega(x, y) &= I(x_i, R_i^*) \cap \Omega(x, y) \\ I(y_i, R_i) \cap \Omega(x, y) &= I(y_i, R_i^*) \cap \Omega(x, y) \\ I(x_i, R_i^*) \cap X_1 &\neq \emptyset \\ I(y_i, R_i^*) \cap X_1 &\neq \emptyset \end{aligned}$$

Proof. Let $R_i \in \mathcal{R}$ and $x, y \in X$ be given. Without loss of generality, assume that $y_i \; R_i \; x_i$. Define $A = I(x_i, R_i) \cap \Omega(x, y)$ and

$$U(x_i, R_i^*) = \bigcap_{a \in A} \left[\bigcap_{q \in C(a, R_i)} H(a, q) \right]$$

where we recall that $H(a, q) = \{b \in \mathbb{R}^\ell_+ \mid q \cdot b \geq q \cdot a\}$. Let $I(x_i, R_i^*)$ be the boundary of $U(x_i, R_i^*)$.

Define a function $g : A \to \mathbb{R}_+$ as follows: For every $a \in A$, if $(a_1 + 1, 0, \ldots, 0) \; P_i \; a$, then let $g(a) = 0$, and otherwise, let $g(a)$ be such that $(a_1 + 1, g(a)a_2, \ldots, g(a)a_\ell) \; I_i \; a$. By strict monotonicity of R_i, $g(a) < 1$. By continuity of R_i, g is continuous. For every $a \in A$, let $b(a) = (a_1 + 1, g(a)a_2, \ldots, g(a)a_\ell)$. Define $f : A \to X_1$ by

$$\begin{aligned} f(a) &= a + \frac{1}{1 - g(a)}[b(a) - a] \\ &= \left(a_1 + \frac{1}{1 - g(a)}, 0, \ldots, 0 \right). \end{aligned}$$

Since $b(a) \; R_i \; a$, it follows that for every $q \in C(a, R_i)$, $q \cdot b(a) \geq q \cdot a$, and so $q \cdot f(a) \geq q \cdot a$. Hence, $f(a) \in H(a, q)$.

The function f is continuous, and the set A is compact and nonempty. Hence, the set $f(A)$ is compact and nonempty. Therefore, there exists $a^* \in A$ such that $||f(a^*)|| = \max_{a \in A} ||f(a)|| = \max_{a \in A} \left[a_1 + \frac{1}{1 - g(a)} \right]$. Then, for all

$a \in A$, and all $q \in C(a, R_i)$, since $f(a) \in H(a,q)$ and $f(a^*) \geq f(a)$, we have $f(a^*) \in H(a,q)$. Thus, $f(a^*) \in U(x_i, R_i^*)$, which proves that $U(x_i, R_i^*) \cap X_1 \neq \emptyset$ and $I(x_i, R_i^*) \cap X_1 \neq \emptyset$.

If $y_i \ I_i \ x_i$, then we are done. Assume that $y_i \ P_i \ x_i$. Define

$$U(y_i, R_i^*) = \bigcap_{a \in I(y_i, R_i) \cap \Omega(x,y)} \left[\bigcap_{q \in C(a, R_i)} H(a,q) \right].$$

There exists $\delta > 0$ such that for all $z_i \in [(1+\delta) I_i(x_i, R_i)] \cap \Omega(x,y)$, $y_i \ P_i \ z_i$. Define

$$\tilde{U}(y_i, R_i^*) = U(y_i, R_i^*) \cap (1+\delta) U(x_i, R_i^*)$$

Then, let $I(y_i, R_i^*)$ be the boundary of $\tilde{U}(y_i, R_i^*)$. Note that $I(x_i, R_i^*) \cap I(y_i, R_i^*) = \emptyset$. A similar argument as above shows that $U(y_i, R_i^*) \cap X_1 \neq \emptyset$. Since $U(x_i, R_i^*) \cap X_1 \neq \emptyset$, we have $[(1+\delta) U(x_i, R_i^*)] \cap X_1 \neq \emptyset$. Thus, $\tilde{U}(y_i, R_i^*) \cap X_1 \neq \emptyset$ and $I(y_i, R_i^*) \cap X_1 \neq \emptyset$. ■

Proof of Lemma 2.

Let $\mathbf{R}, \mathbf{R}' \in \mathcal{R}^n$, $x, y \in X$ be such that for all $i \in N$, R_i and R_i' agree on $\{x_i, y_i\}$, and for no $i \in N$, $x_i \ I_i \ y_i$. Assume that $x \bar{P}(\mathbf{R}) y$.

By Lemma 9, there exists $\mathbf{R}^* \in \mathcal{R}^n$ such that for all $i \in N$,

$$\begin{aligned} I(x_i, R_i) \cap \Omega(x,y) &= I(x_i, R_i^*) \cap \Omega(x,y) \\ I(y_i, R_i) \cap \Omega(x,y) &= I(y_i, R_i^*) \cap \Omega(x,y) \\ I(x_i, R_i^*) \cap X_1 &\neq \emptyset \\ I(y_i, R_i^*) \cap X_1 &\neq \emptyset, \end{aligned}$$

and similarly there exists $\mathbf{R}'^* \in \mathcal{R}^n$ such that for all $i \in N$, ,

$$\begin{aligned} I(x_i, R_i') \cap \Omega(x,y) &= I(x_i, R_i'^*) \cap \Omega(x,y) \\ I(y_i, R_i') \cap \Omega(x,y) &= I(y_i, R_i'^*) \cap \Omega(x,y) \\ I(x_i, R_i'^*) \cap X_2 &\neq \emptyset \\ I(y_i, R_i'^*) \cap X_2 &\neq \emptyset. \end{aligned}$$

Define $x^1, y^1 \in X_1^n$ by $\{x_i^1\} = I(x_i, R_i^*) \cap X_1$ and $\{y_i^1\} = I(y_i, R_i^*) \cap X_1$ for all $i \in N$. Notice that for all $i \in N$, $x_{i1}^1 > 0$, $y_{i1}^1 > 0$ because $x, y \in X$

and preferences are strictly monotonic. Construct $x^{1*}, y^{1*} \in X_1^n$ as follows: for all $i \in N$,

$$\begin{aligned} x_i^{1*} &= x_i^1 + \frac{1}{3}\left|x_i^1 - y_i^1\right| \\ y_i^{1*} &= \max\left\{\frac{1}{2}y_i^1, y_i^1 - \frac{1}{3}\left|x_i^1 - y_i^1\right|\right\}. \end{aligned}$$

Notice that for all $i \in N$,

$$\begin{aligned} x_{i1}^{1*} &> y_{i1}^{1*} \Leftrightarrow x_i P_i y_i \\ y_{i1}^{1*} &> x_{i1}^{1*} \Leftrightarrow y_i P_i x_i. \end{aligned}$$

By Weak Pareto, $x^{1*}\bar{P}(\mathbf{R}^*)x$ and $y\bar{P}(\mathbf{R}^*)y^{1*}$. By IIA-ISFA, $x\bar{P}(\mathbf{R}^*)y$. Therefore, by transitivity,

$$x^{1*}\bar{P}(\mathbf{R}^*)y^{1*}.$$

Now, define $x^2, y^2 \in X_2^n$ by $\{x_i^2\} = I(x_i, R_i'^*) \cap X_2$ and $\{y_i^2\} = I(y_i, R_i'^*) \cap X_2$ for all $i \in N$. Again, $x_{i2}^2 > 0$, $y_{i2}^2 > 0$ for all $i \in N$. Construct $x^{2*}, y^{2*} \in X_2^n$ as follows: for all $i \in N$,

$$\begin{aligned} x_i^{2*} &= \max\left\{\frac{1}{2}x_i^2, x_i^2 - \frac{1}{3}\left|x_i^2 - y_i^2\right|\right\} \\ y_i^{2*} &= y_i^2 + \frac{1}{3}\left|x_i^2 - y_i^2\right|. \end{aligned}$$

Notice that for all $i \in N$,

$$\begin{aligned} x_{i2}^{2*} &> y_{i2}^{2*} \Leftrightarrow x_i P_i' y_i \Leftrightarrow x_i P_i y_i \Leftrightarrow x_{i1}^{1*} > y_{i1}^{1*} \\ y_{i2}^{2*} &> x_{i2}^{2*} \Leftrightarrow y_i P_i' x_i \Leftrightarrow y_i P_i x_i \Leftrightarrow y_{i1}^{1*} > x_{i1}^{1*}. \end{aligned}$$

By Weak Pareto, $x\bar{P}(\mathbf{R}'^*)x^{2*}$ and $y^{2*}\bar{P}(\mathbf{R}'^*)y$.

Let $\mathbf{R}^{**} \in \mathcal{R}^n$ be such that for all $i \in N$,

$$x_i^{2*} P_i^{**} x_i^{1*} \text{ and } y_i^{1*} P_i^{**} y_i^{2*}.$$

Notice that for all $i \in N$,

$$\begin{aligned} I(x_i^{1*}, R_i^{**}) \cap \Omega(x^{1*}, y^{1*}) &= I(x_i^{1*}, R_i^*) \cap \Omega(x^{1*}, y^{1*}) = \{x_i^{1*}\}, \\ I(y_i^{1*}, R_i^{**}) \cap \Omega(x^{1*}, y^{1*}) &= I(y_i^{1*}, R_i^*) \cap \Omega(x^{1*}, y^{1*}) = \{y_i^{1*}\}. \end{aligned}$$

Therefore, by IIA-ISFA, $x^{1*}\bar{P}(\mathbf{R}^{**})y^{1*}$. By Weak Pareto, $x^{2*}\bar{P}(\mathbf{R}^{**})x^{1*}$ and $y^{1*}\bar{P}(\mathbf{R}^{**})y^{2*}$, so that by transitivity, $x^{2*}\bar{P}(\mathbf{R}^{**})y^{2*}$.

Now, we also have that for all $i \in N$,

$$\begin{aligned} I(x_i^{2*}, R_i^{**}) \cap \Omega(x^{2*}, y^{2*}) &= I(x_i^{2*}, R_i'^{*}) \cap \Omega(x^{2*}, y^{2*}) = \{x_i^{2*}\}, \\ I(y_i^{2*}, R_i^{**}) \cap \Omega(x^{2*}, y^{2*}) &= I(y_i^{2*}, R_i'^{*}) \cap \Omega(x^{2*}, y^{2*}) = \{y_i^{2*}\}. \end{aligned}$$

By IIA-ISFA again, $x^{2*}\bar{P}(\mathbf{R}'^{*})y^{2*}$. By transitivity, we deduce $x\bar{P}(\mathbf{R}'^{*})y$. Finally, by IIA-ISFA,

$$x\bar{P}(\mathbf{R}')y.$$

We have proved that $x\bar{P}(\mathbf{R})y$ implies $x\bar{P}(\mathbf{R}')y$,. It follows from symmetry of the argument that $y\bar{P}(\mathbf{R})x$ implies $y\bar{P}(\mathbf{R}')x$, and that $x\bar{I}(\mathbf{R})y$ implies $x\bar{I}(\mathbf{R}')y$. ■

A.3 Proof of Proposition 5

In order to prove the impossibility part, it is convenient to consider various possible sizes of the population. Let $\varepsilon > 0$ and $\lambda \geq 1$ be given. Suppose, to the contrary, that there exists a SOF $\bar{R}$ that satisfies Weak Pareto, IIA-IS[εNλEFA] and Anonymity.

Case $n = 2$. Consider the consumption bundles $x = (10\varepsilon, (2\varepsilon)/(2\lambda), 0, ...)$, $y = (20\varepsilon, (2\varepsilon)/(2\lambda), 0, ...)$, $z = ((2\varepsilon)/(2\lambda), 20\varepsilon, 0, ...)$, $w = ((2\varepsilon)/(2\lambda), 10\varepsilon, 0, ...)$. Let preference relations $R_1 \in \mathcal{R}$ and $R_2 \in \mathcal{R}$ be defined as follows.
(i) On the subset

$$S_1 = \{v \in R_+^\ell \mid \forall i \in \{3, ..., \ell\}, v_i = 0 \text{ and } v_2 \leq \min\{v_1, 2\varepsilon\}\}$$

we have

$$v\ R_1\ v' \Leftrightarrow v_1 + 2v_2 \geq v_1' + 2v_2',$$

and on the subset

$$S_2 = \{v \in R_+^\ell \mid \forall i \in \{3, ..., \ell\}, v_i = 0 \text{ and } v_1 \leq \min\{v_2, 2\varepsilon\}\},$$

we have

$$v\ R_1\ v' \Leftrightarrow 2v_1 + v_2 \geq 2v_1' + v_2'.$$

(ii) On $B_\varepsilon(x) \cup B_\varepsilon(y)$,

$$v \ R_1 \ v' \Leftrightarrow v_1 + 2v_2 + \sum_{k=3}^{\ell} v_k \geq v'_1 + 2v'_2 + \sum_{k=3}^{\ell} v'_k,$$

and on $B_\varepsilon(z) \cup B_\varepsilon(w)$,

$$v \ R_1 \ v' \Leftrightarrow 2v_1 + v_2 + \sum_{k=3}^{\ell} v_k \geq 2v'_1 + v'_2 + \sum_{k=3}^{\ell} v'_k.$$

(iii) Note that the projection of $B_\varepsilon(x) \cup B_\varepsilon(y)$ on the subspace of good 1 and good 2, namely, $[B_\varepsilon(x) \cup B_\varepsilon(y)] \cap \{v \in R_+^\ell : | : \forall i \in \{3, ..., \ell\}, v_i = 0\}$, is included in S_1, and the projection of $B_\varepsilon(z) \cup B_\varepsilon(w)$ on the subspace of good 1 and good 2 is included in S_2. Since

$$[w_1 + (2\varepsilon - w_1)] + 2\left[w_2 - 2\left(2\varepsilon - w_1\right)\right] > x_1 + 2x_2$$

and

$$2\left[y_1 - 2\left(2\varepsilon - y_2\right)\right] + \left[y_2 + \left(2\varepsilon - y_2\right)\right] > 2z_1 + z_2,$$

it is possible to complete the definition of R_1 so that $w \ P_1 \ x$ and $y \ P_1 \ z$. Then, define R_2 so that it coincides with R_1 on S_1, on S_2, and on $B_\varepsilon(a)$ for all $a \in \{x, y, z, w\}$. Similarly, it is possible to complete the definition of R_2 so that $x \ P_2 \ w$ and $z \ P_2 \ y$. Figure 2 illustrates this construction (for $\lambda = 1$).

If the profile of preferences is $\mathbf{R} = (R_1, R_2)$, by Weak Pareto we have

$$(y, x)\bar{P}(\mathbf{R})(z, w) \text{ and } (w, z)\bar{P}(\mathbf{R})(x, y).$$

If the profile of preferences is $\mathbf{R}' = (R_1, R_1)$, by Anonymity we have

$$(y, x)\bar{I}(\mathbf{R}')(x, y) \text{ and } (w, z)\bar{I}(\mathbf{R}')(z, w).$$

Notice that $\lambda\Omega(x, y) \subseteq S_1 \cup S_2$ and $\lambda\Omega(z, w) \subseteq S_1 \cup S_2$. Since R_1 and R_2 coincide on S_1, on S_2, and on $B_\varepsilon(a)$ for all $a \in \{x, y, z, w\}$, it follows from IIA-IS[εNλEFA] that

$$\begin{array}{rcl} (y, x)\bar{I}(\mathbf{R}')(x, y) & \Leftrightarrow & (y, x)\bar{I}(\mathbf{R})(x, y) \\ \text{and } \ (w, z)\bar{I}(\mathbf{R}')(z, w) & \Leftrightarrow & (w, z)\bar{I}(\mathbf{R})(z, w). \end{array}$$

By transitivity, $(x, y)\bar{P}(\mathbf{R})(x, y)$, which is impossible.

Case $n = 3$. Consider the consumption bundles $x = (10\varepsilon, (2\varepsilon)/(3\lambda), 0, ...)$, $y = (20\varepsilon, (2\varepsilon)/(3\lambda), 0, ...)$, $t = (15\varepsilon, (2\varepsilon)/(3\lambda), 0, ...)$, $z = ((2\varepsilon)/(3\lambda), 20\varepsilon, 0, ...)$, $w = ((2\varepsilon)/(3\lambda), 10\varepsilon, 0, ...)$, $r = ((2\varepsilon)/(3\lambda), 15\varepsilon, 0, ...)$. Let preference relations R_1, R_2 and R_3 be defined as above on the subset S_1, on S_2, and on $B_\varepsilon(a)$ for all $a \in \{x, y, z, w, t, r\}$. And complete their definitions so that yP_1z, wP_1x, tP_2r, zP_2y, xP_3w, rP_3t.

If the profile of preferences is $\mathbf{R} = (R_1, R_2, R_3)$, by Weak Pareto we have

$$(y, t, x)\bar{P}(\mathbf{R})(z, r, w) \text{ and } (w, z, r)\bar{P}(\mathbf{R})(x, y, t).$$

If the profile of preferences is $\mathbf{R}' = (R_1, R_1, R_1)$, by Anonymity we have

$$(y, t, x)\bar{I}(\mathbf{R}')(x, y, t) \text{ and } (w, z, r)\bar{I}(\mathbf{R}')(z, r, w).$$

Notice that $\lambda\Omega(x, y, t) \subseteq S_1 \cup S_2$ and $\lambda\Omega(z, w, r) \subseteq S_1 \cup S_2$. Since R_1, R_2 and R_3 coincide on S_1, on S_2, and on $B_\varepsilon(a)$ for all $a \in \{x, y, t, z, w, r\}$, it follows from IIA-IS[εNλEFA] that

$$\begin{aligned} (y, t, x)\bar{I}(\mathbf{R}')(x, y, t) &\Leftrightarrow (y, t, x)\bar{I}(\mathbf{R})(x, y, t) \\ \text{and } (w, z, r)\bar{I}(\mathbf{R}')(z, r, w) &\Leftrightarrow (w, z, r)\bar{I}(\mathbf{R})(z, r, w). \end{aligned}$$

By transitivity, $(x, y, t)\bar{P}(\mathbf{R})(x, y, t)$, which is impossible.

Case $n = 2k$. Partition the population into k pairs, and construct an argument similar to the case $n = 2$, with the consumption bundles $x = (10\varepsilon, (2\varepsilon)/(n\lambda), 0, ...)$, $y = (20\varepsilon, (2\varepsilon)/(n\lambda), 0, ...)$, $z = ((2\varepsilon)/(n\lambda), 20\varepsilon, 0, ...)$, $w = ((2\varepsilon)/(n\lambda), 10\varepsilon, 0, ...)$, and the allocations $(y, x, y, x, ...)$, $(x, y, x, y, ...)$, $(z, w, z, w, ...)$ and $(w, z, w, z, ...)$.

Case $n = 2k + 1$. Partition the population into $k - 1$ pairs and one triple, and construct an argument combining the cases $n = 2$ and $n = 3$, with the consumption bundles $x = (10\varepsilon, (2\varepsilon)/(n\lambda), 0, ...)$, $y = (20\varepsilon, (2\varepsilon)/(n\lambda), 0, ...)$, $t = (15\varepsilon, (2\varepsilon)/(n\lambda), 0, ...)$, $z = ((2\varepsilon)/(n\lambda), 20\varepsilon, 0, ...)$, $w = ((2\varepsilon)/(n\lambda), 10\varepsilon, 0, ...)$, $r = ((2\varepsilon)/(n\lambda), 15\varepsilon, 0, ...)$, and the allocations $(y, x, y, x, ..., y, t, x)$, $(x, y, x, y, ..., x, y, t)$, $(z, w, z, w, ...z, r, w)$ and $(w, z, w, z, ..., w, z, r)$. ■

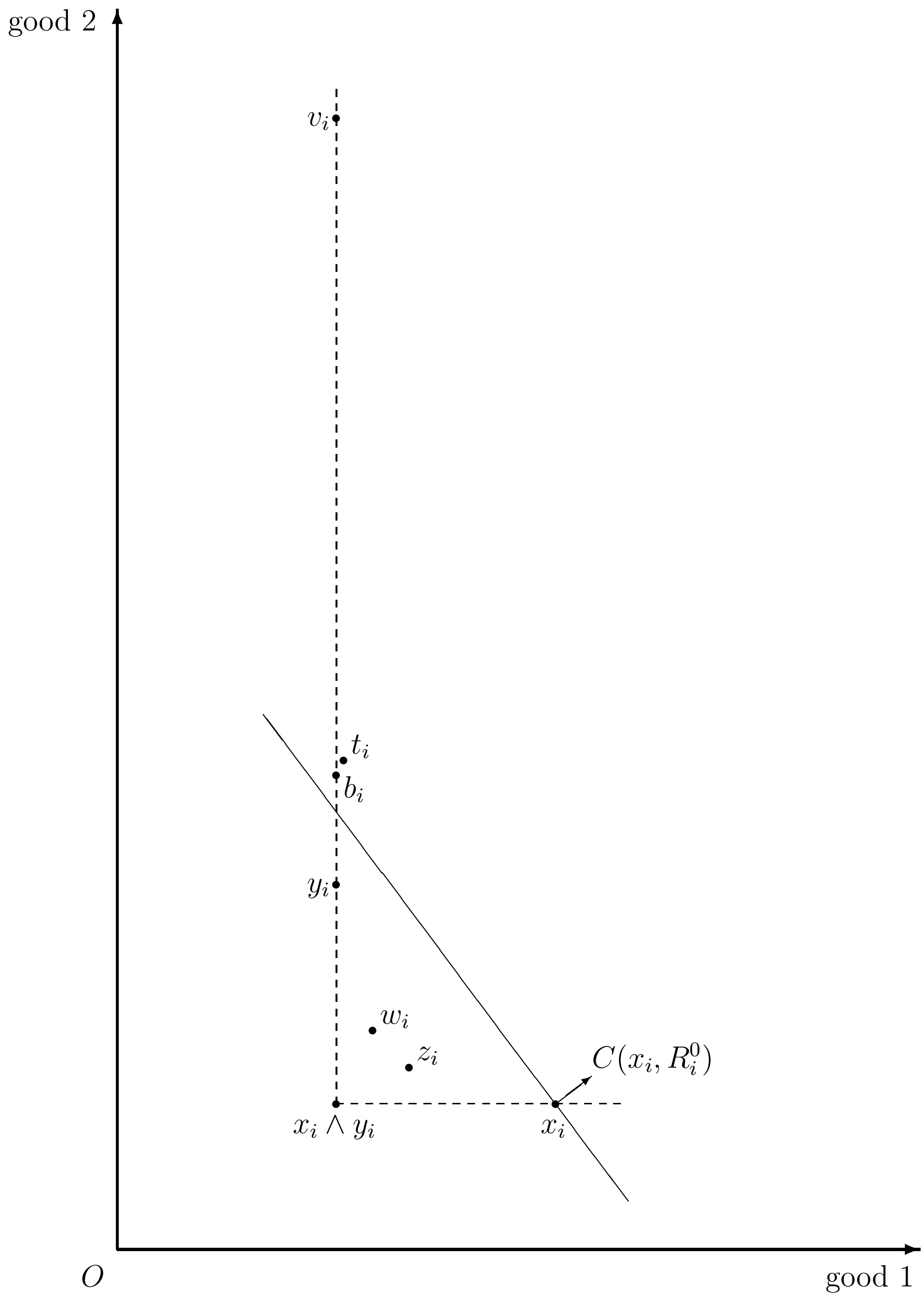
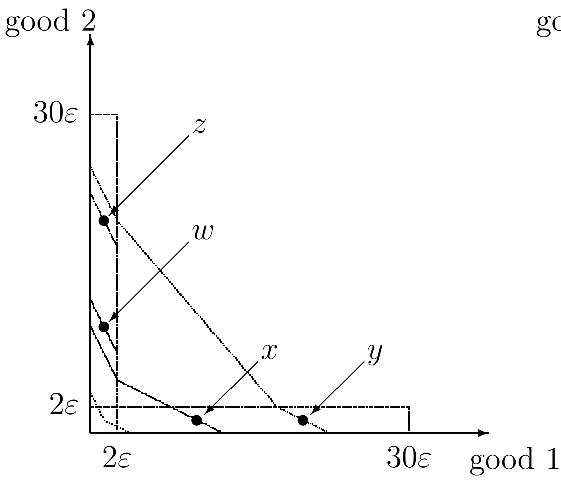


Figure 1: Proof of Lemma 7



Preferences R_1 Preferences R_2

Figure 2: Proof of Proposition 5