The Fundamental Theorems of Welfare Economics in a Non-Welfaristic Approach*

Koichi Tadenuma
Faculty of Economics, Hitotsubashi University
Kunitachi, Tokyo 186-8601, Japan
Email: tadenuma@econ.hit-u.ac.jp

Yongsheng Xu
Department of Economics
Andrew Young School of Policy Studies
Georgia State University, Atlanta, GA, 30303, U.S.A.
Email: yxu3@gsu.edu

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Abstract

This paper investigates extensions of the two fundamental theorems of welfare economics to the framework in which each agent is endowed with three types of preference relations: an allocation preference relation, an opportunity preference relation, and an overall preference relation. It is shown that, under certain conditions, the two theorems can be extended. It is also pointed out that the conditions underlying the positive results are restrictive.
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1 Introduction

It is often argued that the market mechanism promotes individual freedom and makes individuals free to choose. However, in the traditional framework of economic theory, the market mechanism is evaluated exclusively on the basis of its allocation efficiency. The notion of allocation efficiency is typically a "welfaristic" notion in which final allocations resulting from the market mechanism are judged by the welfare levels of the individuals involved. Nothing can be said about freedom given to each individual.

Recently, Amartya Sen (1993) put forth explicitly an argument for the market mechanism to promote individual freedom and to make individuals free to choose. He distinguished two aspects of freedom: "the opportunity aspect" and "the process aspect". The opportunity aspect relates to the opportunities of achieving things that each individual values, while the process aspect is concerned with free decisions of each individual. Sen also established that, under certain type of assessments of opportunities, the market mechanism attains efficient states in terms of opportunity-freedom. Following Sen (1993), formal frameworks for "non-welfaristic" analyses have been developed by Suzumura and Xu (2000, 2001) and by Tadenuma and Xu (2001). In this paper, using the framework as in Tadenuma and Xu (2001), we examine the performance of the market mechanism from a non-welfaristic perspective by paying particular attention to opportunity-efficiency, the notion of efficiency in terms of the distributions of opportunity sets, and to overall-efficiency, the notion of efficiency in terms of the distributions of pairs of an opportunity set and a consumption bundle.

In the traditional framework of welfare economics, the performance of market mechanisms is best summarized by the two fundamental theorems: (i) *the first theorem*, which claims that, under certain conditions, a market mechanism generates an efficient allocation, and (ii) *the second theorem*, which asserts that, under some more restrictive conditions, any efficient allocation can be achieved by a market mechanism through an appropriately redistribution of agents' initial endowments. Note that in both theorems, the notion of efficiency is based solely on individual preferences over final consumption bundles, and is a welfaristic one in nature.

To shed light on freedom aspects of a market mechanism, we need to expand our framework in general and to go beyond the usual notion of efficiency in particular. For this purpose, we define a *configuration* as a pair of an allocation and a distribution of opportunity sets. While the usual notion of efficiency is based only on the allocation part of a configuration, a configuration contains information about the distribution of opportunity sets as well. An agent's opportunity set is viewed as reflecting the opportunity aspect of freedom (see, for example, Sen (1988, 1993, 2000) and Pattanaik and Xu (1990)). Depending on how each agent ranks his opportunity sets as well as on how he assesses his pairs of consumption bundles and opportunity sets, we can go beyond the notion of allocation efficiency to the notion of efficiency with respect to distributions of opportunity sets reflecting freedom aspects, and to the notion of efficiency with respect to configurations reflecting agents' overall attitudes toward consumption bundles and opportunities.

To formalize these ideas, we assume that each agent is endowed with three types of preference relations: an allocation preference relation that ranks consumption bundles, an opportunity preference relation that ranks opportunity sets, and an overall preference relation that ranks pairs of consumption bundles and the associated opportunity sets. Using above preference relations, we introduce three efficiency conditions: (i) *allocation-Pareto-optimality*, which is the usual notion of efficiency of allocations, (ii) *opportunity-Pareto-optimality*, which reflects the situation in which it is impossible to improve one agnet's opportunities without reducing any other agent's opportunities, and (iii) *overall-Pareto-optimality*, which concerns the possibility of improving one agent's overall situation without hurting any other agent's overall situation. See Section 3 for formal definitions of these concepts.

With these concepts in hand, we examine the relationship between the market mechanism and various notions of Pareto optimality, with particular concern of extending the two fundamental theorems of welfare economics. We show that, if the opportunity-preference relation of every agent belongs to the class of opportunity-preference relations that is discussed and examined by Sen (1993), then the two welfare theorems can be extended without much difficulty. However, once we venture out of this class of opportunity-preference relations, we may no longer be able to extend the two welfare theorems. Therefore, the conviction that market mechanisms promote individual opportunity-freedom is valid in some limited cases but is not true in general.

The organization of the remainder of the paper is as follows. In Section 2, we present some basic notation and definitions. Section 3 introduces several notions of Pareto optimality in our framework. Extensions of the first welfare theorem and the second theorem are examined in Sections 4 and 5, respectively. Section 6 discusses the restrictive nature of our results. We offer some concluding remarks in Section 7.

2 Notation and Definitions

There are n agents and k goods. Let $N = \{1, \ldots, n\}$ be the set of agents. An *allocation* is a vector $x = (x_1, \ldots, x_n) \in \mathbb{R}^{nk}_+$ where for each $i \in N$, $x_i = (x_{i1}, \ldots, x_{ik}) \in \mathbb{R}^k_+$ is a *consumption bundle* of agent i.[1] There exist some fixed amounts of social endowments of goods, which are represented by the vector $\omega \in \mathbb{R}^k_{++}$. An allocation $x \in \mathbb{R}^{nk}_+$ is *feasible* if $\sum_{i=1}^n x_i \leq \omega$.[2] Let Z be the set of all feasible allocations.

For each $i \in N$, an *opportunity set* for agent i is a set in $\mathbb{R}^k_+$. In this paper, we consider opportunity sets that are *compact* and *comprehensive*.[3] The class of all compact and comprehensive opportunity sets are denoted by $\mathcal{O}$. A special subclass of $\mathcal{O}$, namely the class of (constrained) budget sets, has some importance in this paper. Let $\Omega \equiv \{y_0 \in \mathbb{R}^k_+ \mid y_0 \leq \omega\}$. For each $i \in N$, a *budget set* for agent i at a price vector $p \in \mathbb{R}^k_+$ and a consumption bundle $x_i \in \mathbb{R}^k_+$ is defined by

$$B(p, x_i) = \{y_i \in \mathbb{R}^k_+ \mid p \cdot y_i \leq p \cdot x_i\} \cap \Omega$$

Let $\mathcal{B} = \{B(p, x_i) \mid p \in \mathbb{R}^k_+, x_i \in \mathbb{R}^k_+\}$. A *distribution of opportunity sets* is an n-tuple $O = (O_1, \ldots, O_n) \in \mathcal{O}^n$. A *configuration* is a pair $(x, O) \in \mathbb{R}^{nk}_+ \times \mathcal{O}^n$ such that $x_i \in O_i$ for all $i \in N$. For each $(x, O) \in \mathbb{R}^{nk}_+ \times \mathcal{O}^n$ and each $i \in N$, the *individual state of agent i at* (x, O) is the pair (x_i, O_i).

For each $i \in N$ and each $O_i \in \mathcal{O}$, let

$$\partial O_i \equiv \{y_i \in O_i \mid \forall w_i \in \mathbb{R}^k_+ : w_i >> y_i \Rightarrow w_i \notin O_i\}$$

A configuration $(x, O) \in \mathbb{R}^{nk}_+ \times \mathcal{O}^n$ is *feasible* if (i) $x \in Z$ and (ii) for every $i \in N$, and every $y_i \in \partial O_i$, there exists $(y_1, \ldots, y_{i-1}, y_{i+1}, \ldots, y_n) \in \Pi_{j \neq i} \partial O_j$

[1] As usual, $\mathbb{R}_+$ is the set of all non-negative real numbers, and $\mathbb{R}_{++}$ is the set of all positive real numbers.

[2] Vector inequalities are as usual: $\geq, >$ and $>>$.

[3] A set $A \subseteq \mathbb{R}^k_+$ is *comprehensive* if $x \in A$ and $0 \leq y \leq x$ imply $y \in A$. Comprehensiveness is a reasonable assumption for opportunity sets. It means that if a consumption bundle x is available for an agent, then any consumption bundle y containing a less amount of each good than x is also available for the agent.

such that $(y_1, \ldots, y_n) \in Z$.[4]

Given a set X, a *preference quasiorder* on X is a reflexive and transitive binary relation on X. When a preference quasiorder is also complete, it is called a *preference order*. Each agent $i \in N$ is endowed with the following three preference relations.

1. An *allocation preference order* on $\mathbb{R}^k_+$, denoted R^A_i, which is continuous and monotonic.
2. An *opportunity preference quasiorder* on $\mathcal{O}$, denoted R^O_i, which is *monotonic* in the following sense:
(i) For all $O_1, O_2 \in \mathcal{O}, O_1 \subseteq O_2 \Rightarrow O_2 \; R^O_i \; O_1$
(ii) For all $O_1, O_2 \in \mathcal{O}, O_1 \subset \text{int}O_2 \Rightarrow O_2 \; P^O_i \; O_1$, where $\text{int}O_2$ is the relative interior of O_2 in $\mathbb{R}^k_+$.[5]
3. An *overall preference quasiorder* on $\mathbb{R}^k_+ \times \mathcal{O}$, denoted $\bar{R}_i$.

Let $\mathcal{R}^A, \mathcal{R}^O$, and $\bar{\mathcal{R}}$ denote the classes of allocation preference orders, opportunity preference quasiorders, and overall preference quasiorders, respectively. Let $\mathcal{R} = \mathcal{R}^A \times \mathcal{R}^O \times \bar{\mathcal{R}}$. A *preference profile* is a list $R = (R_1, \ldots, R_n) \in \mathcal{R}^n$ where $R_i = (R^A_i, R^O_i, \bar{R}_i)$ for each $i \in N$.

We will consider several conditions on the relationships between overall preference quasiorders and the other two preference quasiorders:

Condition A: For all $(x_i, O_i), (y_i, C_i) \in \mathbb{R}^k_+ \times \mathcal{O}$,
(i) $(x_i, O_i) \; \bar{R}_i \; (y_i, C_i) \Rightarrow [x_i \; R^A_i \; y_i$ or $O_i \; R^O_i \; C_i]$,
(ii) $(x_i, O_i) \; \bar{P}_i \; (y_i, C_i) \Rightarrow [x_i \; P^A_i \; y_i$ or $O_i \; P^O_i \; C_i]$.

Condition B:[6] For all $(x_i, O_i), (y_i, C_i) \in \mathbb{R}^k_+ \times \mathcal{O}$,
(i) $[x_i \; R^A_i \; y_i$ and $O_i \; R^O_i \; C_i] \Rightarrow (x_i, O_i) \; \bar{R}_i \; (y_i, C_i)$,
(ii) $[x_i \; P^A_i \; y_i$ and $O_i \; P^O_i \; C_i] \Rightarrow (x_i, O_i) \; \bar{P}_i \; (y_i, C_i)$.

Note that in general, there is no logical relation between Conditions A and B. However, if the opportunity preference quasiorder is complete, then Condition B implies Condition A, whereas if the overall preference quasiorder

[4] Our notion of a feasible distribution of opportunity sets as figured in the definition of a feasible configuration reflects the idea that, in a social setting, one agent's opportunity set may depend on other agents' opportunity sets, and therefore, agents' opportunity-freedoms are inter-related to and inter-dependent on each other. See, for example, Basu (1987), Gravel, Laslier and Trannoy (1998), and Pattanaik (1994) for some discussions of the issues relating to the inter-dependence of agents' opportunity sets and freedom of choice.

[5] As usual, the strict preference relation associated to R^O_i is denoted P^O_i. Similar notation is used for other preference relations.

[6] In Tadenuma and Xu (2001), we assume this condition.

is complete, then Condition A implies Condition B. (Hence, if both the opportunity preference quasiorder and the overall preference quasiorder are complete, then Conditions A and B are equivalent.)

We will also consider some "extreme" preferences. Let $R_i = (R_i^A, R_i^O, \bar{R}_i) \in \mathcal{R}$ be given. We say that agent $i \in N$ is a *consequentialist at* R_i if $\forall (x_i, B_i), (y_i, C_i) \in \mathbb{R}_+^k \times \mathcal{O} : (x_i, B_i)\bar{R}_i(y_i, C_i) \Leftrightarrow x_i\ R_i^A\ y_i$. We say that agent $i \in N$ is a *non-consequentialist at* R_i if $\forall (x_i, B_i), (y_i, C_i) \in \mathbb{R}_+^k \times \mathcal{O} : (x_i, B_i)\bar{R}_i(y_i, C_i) \Leftrightarrow B_i\ R_i^O\ C_i$.

3 Pareto-Optimality

In this paper, we focus on "autonomous" configurations, at which the specified consumption bundle for each agent is actually the best one in his opportunity set by his allocation preference order.

Let $R = (R_1, \ldots, R_n) \in \mathcal{R}^n$ be given, where $R_i = (R_i^A, R_i^O, \bar{R}_i)$ for all $i \in N$. We say that a configuration $(x, O) \in \mathbb{R}_+^k \times \mathcal{O}$ is *autonomous* for R if it is feasible and for every $i \in N$, and every $y_i \in B_i$, $x_i\ R_i^A\ y_i$. Let $\mathbb{Z}(R)$ denote the set of all autonomous configurations for R. A configuration $(x, B) \in \mathbb{Z}$ is *decentralizable* for R if it is autonomous and $B \in \mathcal{B}^n$. Let $\mathbb{D}(R)$ be the set of decentralizable configurations for R.

A configuration $(y, C) \in \mathbb{Z}$ *allocation-Pareto-dominates* $(x, O) \in \mathbb{Z}$ for R if $y_i\ R_i^A\ x_i$ for every $i \in N$ and $y_i\ P_i^A\ x_i$ for some $i \in N$. A configuration $(x, O) \in \mathbb{Z}$ is *allocation-Pareto-optimal* for R if no *autonomous* configuration allocation-Pareto-dominates it. We define *opportunity-Pareto-domination* and *opportunity-Pareto-optimality* by replacing allocation preference orders with opportunity preference quasiorders in the above definitions. We also define analogously *overall-Pareto-domination* and *overall-Pareto-optimality* based on overall preference quasiorders.

Sen (1993) considers the following conditions of opportunity preference quasiorders (See Sen's Axiom O on p.530):
(i) for all $O_i, C_i \in \mathcal{O}$, $O_i\ R_i^O\ C_i$ *only if* there exists $x_i \in O_i$ such that $x_i\ R_i^A\ y_i$ for every $y_i \in C_i$, and
(ii) for all $O_i, C_i \in \mathcal{O}$, $O_i\ P_i^O\ C_i$ *only if* there exists $x_i \in O_i$ such that $x_i\ P_i^A\ y_i$ for all $y_i \in C_i$.
Let $\hat{\mathcal{R}}^O$ be the class of opportunity preference quasiorders satisfying the above conditions. Note that the above conditions are *necessary conditions* for $O_i\ R_i^O\ C_i$ and $O_i\ P_i^O\ C_i$. However, if R_i^O is complete, then they are also

sufficient conditions.

When $R_i^O \in \hat{\mathcal{R}}^O$ for every $i \in N$, several logical relations hold between various notions of Pareto-optimality in our framework.

Proposition 1 *Let $R = (R_1, \ldots, R_n) \in \mathcal{R}^n$ be such that for every $i \in N$, $R_i^O \in \hat{\mathcal{R}}^O$.*

(i) *If $(x, O) \in \mathbb{Z}(R)$ is allocation-Pareto-optimal for R, then (x, O) is opportunity-Pareto-optimal for R.*

(ii) *Suppose that for every $i \in N$, R_i^O is complete. Then, $(x, O) \in \mathbb{Z}(R)$ is allocation-Pareto-optimal for R if and only if (x, O) is opportunity-Pareto-optimal for R.*

Proof:
(i) Let $(x, O) \in \mathbb{Z}(R)$ be an allocation-Pareto-optimal configuration for R. Suppose, to the contrary, that there exists $(y, C) \in \mathbb{Z}(R)$ that opportunity-Pareto-dominates (x, O). Then, for every $i \in N$, $C_i \ R_i^O \ O_i$, and there exists $j \in N$ with $C_j \ P_j^O \ O_j$. For every $i \in N$, since $R_i^O \in \hat{\mathcal{R}}^O$, there exists $z_i \in C_i$ such that $z_i \ R_i^A \ w_i$ for all $w_i \in O_i$. Because (y, C) is autonomous and $z_i \in C_i$, we have $y_i \ R_i^A \ z_i$. By the transitivity of R_i^A, $y_i R_i^A w_i$ for all $w_i \in O_i$. Letting $w_i = x_i$, we have $y_i R_i^A x_i$. Similarly, we can show that $y_j \ P_j^A \ x_j$. Thus, (y, C) allocation-Pareto-dominates (x, O), which contradicts the allocation-Pareto-optimality of (x, O). Therefore, (x, O) is opportunty-Pareto-optimal.
(ii) Suppose that for every $i \in N$, R_i^O is complete. Let $(x, O) \in \mathbb{Z}(R)$. If (x, O) is allocation-Pareto-optimal for R, then, from (i), (x, O) is opportunity-Pareto-optimal for R. It remains to show that if (x, O) is opportunity-Pareto-optimal for R, then (x, O) is allocation-Pareto-optimal for R as well. Let (x, O) be opportunity-Pareto-optimal for R. Suppose, to the contrary, that there exists $(y, C) \in \mathbb{Z}(R)$ that allocation-Pareto-dominates (x, O). Then, for every $i \in N$, $y_i \ R_i^A \ x_i$, and there exists $j \in N$ with $y_j \ P_j^A \ x_j$. Since (x, O) is autonomous, it follows that for every $i \in N$, and every $w_i \in O_i$, $x_i \ R_i^A \ w_i$, and hence, $y_i R_i^A w_i$. Because $R_i^O \in \hat{\mathcal{R}}^O$ for every $i \in N$, there exists no $i \in N$ with $O_i \ P_i^O \ C_i$. Since R_i^O is complete, we have $C_i \ R_i^O \ O_i$ for every $i \in N$. By a similar argument, we can show that $C_j \ P_j^O \ O_j$. Hence, (y, C) opportunity-Pareto-dominates (x, O), which is a contradiction. Therfore, (x, O) is allocation-Pareto-optimal. ■

Proposition 2 *Let $R = (R_1, \ldots, R_n) \in \mathcal{R}^n$ be such that for every $i \in N$, $R_i^O \in \hat{\mathcal{R}}^O$.*

(i) *Suppose that for every $i \in N$, R_i satisfies Condition A. If $(x, O) \in \mathbb{Z}(R)$ is allocation-Pareto-optimal for R, then (x, O) is overall-Pareto-optimal for R.*

(ii) *Suppose for every $i \in N$, R_i satisfies Condition B. If $(x, O) \in \mathbb{Z}(R)$ is overall-Pareto-optimal for R, then (x, O) is allocation-Pareto-optimal for R.*

Proof:
(i) Suppose that for every $i \in N$, R_i satisfies Condition A. Let $(x, O) \in \mathbb{Z}(R)$ be an allocation-Pareto-optimal configuration for R. Suppose, to the contrary, that there exists $(y, C) \in \mathbb{Z}(R)$ that overall-Pareto-dominates (x, O). Then, for every $i \in N$, $(y_i, C_i)\bar{R}_i(x_i, O_i)$, and there exists $j \in N$ with $(y_j, C_j)\bar{P}_j(x_j, O_j)$. Because for every $i \in N$, R_i satisfies Condition A, we have $[y_i \ R_i^A \ x_i$ or $C_i \ R_i^O \ O_i]$ for every $i \in N$, and $[y_j \ P_j^A \ x_j$ or $C_j \ P_j^O \ O_j]$. Since $R_i^O \in \hat{\mathcal{R}}^O$ for every $i \in N$, and (y, C) is autonomous for R, it follows that for every $i \in N$, $[C_i \ R_i^O \ O_i \Rightarrow y_i \ R_i^A \ x_i]$, and $[C_i \ P_i^O \ O_i \Rightarrow y_i \ P_i^A \ x_i]$. Therefore, for every $i \in N, y_i \ R_i^A \ x_i$, and $y_j \ P_j^A \ x_j$, which contradicts the allocation-Pareto-optimality of (x, O). Therefore, (x, O) is overall-Pareto-optimal.
(ii) Suppose that for every $i \in N$, R_i satisfies Condition B. Let $(x, O) \in \mathbb{Z}(R)$ be an overall-Pareto-optimal configuration for R. Suppose, to the contrary, that there exists $(y, C) \in \mathbb{Z}(R)$ that allocation-Pareto-dominates (x, O). Then, for all $i \in N$, $y_i \ R_i^A \ x_i$, and there exists $j \in N$ with $y_j \ P_j^A \ x_j$. By the same argument as in the proof of Proposition 1 (ii), we can show that $C_i \ R_i^O \ O_i$ for every $i \in N$, and $C_j \ P_j^O \ O_j$. Since R_i satisfies Condition B, it follows that $(y_i, C_i)\bar{R}_i(x_i, O_i)$ for every $i \in N$, and $(y_j, C_j)\bar{P}_j(x_j, O_j)$. Hence, (y, C) overall-Pareto-dominates (x, O), which is a contradiction. Thus, (x, O) is allocation-Pareto-optimal. ■

Corollary 1 *Let $R = (R_1, \ldots, R_n) \in \mathcal{R}^n$ be such that for every $i \in N$, $R_i^O \in \hat{\mathcal{R}}^O$, R_i^O is complete, and R_i satisfies Condition B. Let $(x, O) \in \mathbb{Z}(R)$. Then, the following three statements are equivalent:*
(a) (x, O) *is allocation-Pareto-optimal for R,*
(b) (x, O) *is opportunity-Pareto-optimal for R,*
(c) (x, O) *is overall-Pareto-optimal for R.*

Proof: When R_i^O is complete for every $i \in N$, the equivalence of (a) and (b) follows from Proposition 1(ii). From our remarks immediately after introducing Conditions A and B, when R_i^O is complete, Condition B implies Condition A. Then, the equivalence of (a) and (c) follows from Proposition 2. ■

4 Extensions of the First Welfare Theorem

To begin with, we observe the following results that do not rely on any further assumptions on opportunity-preference quasiorders, and that follow from our assumptions on allocation-preference orders.

Proposition 3 *For every $R \in \mathcal{R}^n$, every decentralizable configuration for R is allocation-Pareto-optimal for R.*

Proposition 4 *Let $R \in \mathcal{R}^n$ be such that every $i \in N$ is a consequentialist at R. Then, every decentralizable configuration for R is overall-Pareto-optimal for R.*

Proposition 3 is a restatement of the classical first welfare theorem in our framework. Proposition 4 reiterates that, in a consequentialist framework, the extension of the first welfare theorem is straightforward.

Next, we consider extensions of the first welfare theorem when the opportunity-preference quasiorder of each agent is in the class $\hat{\mathcal{R}}^O$. The following proposition summarizes our findings.

Proposition 5 *Let $R = (R_1, \ldots, R_n) \in \mathcal{R}^n$ be such that for every $i \in N$, $R_i^O \in \hat{\mathcal{R}}^O$.*

(i) [Sen, 1993, and Tadenuma and Xu, 2001] *Every decentralizable configuration for R is opportunity-Pareto-optimal for R.*

(ii) *Suppose that for every $i \in N$, i is a non-consequentialist at R_i. Then, every decentralizable configuration for R is overall-Pareto-optimal for R.*

(iii) *Suppose that for every $i \in N$, R_i satisfies Condition A. Then, every decentralizable configuration for R is overall-Pareto-optimal for R.*

Proof: Claim (i) follows from Propositions 1(i) and 3. Claim (ii) follows from Claim (i) and the defintion of a non-consequentialist, and Claim (iii) follows from Propostions 2(i) and 3. ■

Therefore, if each agent's opportunity-preference quasiorder is the type discussed by Sen (1993), there is *essentially* no difficulty of extending the classical first welfare theorem to our framework. Note that the type of an agent's opportunity-preference quasiorders discussed by Sen is intimately linked with the agent's allocation-preference order. Together with Condition A, the agent comes very close being a consequentialist. It is therefore not surprising to see that the first welfare theorem can be extended in the current framework without much difficulty.

5 Extensions of the Second Welfare Theorem

In this section, we assume that for every $i \in N$, the allocation-preference order $R_i^A \in \mathcal{R}^A$ is convex. When the opportunity preference quasiorder of every agent is complete, we obtain the following results.

Proposition 6 *Let $R = (R_1, \ldots, R_n) \in \mathcal{R}^n$ be such that for every $i \in N$, $R_i^O \in \hat{\mathcal{R}}^O$, and R_i^O is complete. Let $(x, O) \in \mathbb{Z}(R)$ with $x_i >> 0$ for every $i \in N$. If (x, O) is either allocation-Pareto-optimal or opportunity-Pareto-optimal for R, then there exists a decentralizable configuration $(x, B) \in \mathbb{D}(R)$ such that $B_i\ I_i^O\ O_i$ for every $i \in N$.*

Proof: We note that, by Proposition 1(ii), when R_i^O is complete for every $i \in N$, (x, B) is allocation-Pareto-optimal for R if and only if it is opportunity-Pareto-optimal for R. Suppose that $x_i >> 0$ for every $i \in N$, and (x, O) is allocation-Pareto-optimal for R (and hence opportunity-Pareto-optimal for R as well.) Since for every $i \in N$, R_i^A is continuous, monotonic and convex, and $x_i >> 0$ for all $i \in N$, it follows from the second fundamental theorem of welfare economics that there exists a price vector $p \in \mathbb{R}_+^k$ such that $(x, B) \in \mathbb{D}(R)$ with $B \equiv (B(p, x_1), \ldots, B(p, x_n))$. To show that $B_i\ I_i^O\ O_i$ for every $i \in N$, suppose the contrary. Then, since R_i^O is complete for every $i \in N$, and (x, O) is opportunity-Pareto-optimal for R, there must exist $j \in N$ with $O_j\ P_j^O\ B_j$. Because $R_j^O \in \hat{\mathcal{R}}^O$ and (x, O) is autonomous for R, $x_j\ P_j^A\ w_j$ for every $w_j \in B_j$. But then, since $x_j \in B_j$, we have $x_j\ P_j^A\ x_j$, which is a contradiction. Thus, for every $i \in N$, $B_i\ I_i^O\ O_i$. ■

Proposition 7 *Let* $R = (R_1, \ldots, R_n) \in \mathcal{R}^n$ *be such that for every* $i \in N$, $R_i^O \in \hat{\mathcal{R}}^O$, R_i^O *is complete, and* R_i *satisfies Condition B. Let* $(x, O) \in \mathbb{Z}(R)$ *with* $x_i >> 0$ *for* $i \in N$. *Suppose that one of the following three statements is true:*
(a) (x, O) *is allocation-Pareto optimal for* R.
(b) (x, O) *is opportunity-Pareto optimal for* R.
(c) (x, O) *is overall-Pareto optimal for* R.
Then, there exists a decentralizable configuration $(x, B) \in \mathbb{D}(R)$ *such that for every* $i \in N$, $B_i\ I_i^O\ O_i$ *and* $(x_i, B_i)\bar{I}_i(x_i, O_i)$.

Proof: Let $(x, O) \in \mathbb{Z}(R)$ be such that $x_i >> 0$ for every $i \in N$.
(a): Suppose that (x, O) is allocation-Pareto-optimal for R. From Proposition 6, there exists a decentralizable configuration $(x, B) \in \mathbb{D}(R)$ such that $B_i\ I_i^O\ O_i$ for every $i \in N$. Obviously, for every $i \in N$, $x_i\ I_i^A\ x_i$. Then, since R_i satisfies Condition B, it follows that $(x_i, O_i)\bar{I}_i(x_i, B_i)$ for every $i \in N$.
(b),(c): Suppose that (x, O) is opportunity-Pareto-optimal or overall-Pareto-optimal for R. Then, by Propositions 1 and 2, (x, O) is allocation-Pareto-optimal for R. Hence, the claim follows from the argument in (a). ■

If we do not require completeness of opportunity preference relations, then we have the following results.

Proposition 8 *Let* $R = (R_1, \ldots, R_n) \in \mathcal{R}^n$ *be such that for every* $i \in N, R_i^O \in \hat{\mathcal{R}}^O$. *Let* $(x, O) \in \mathbb{Z}(R)$ *with* $x_i >> 0$ *for every* $i \in N$. *If* (x, O) *is allocation-Pareto optimal for* R*, then there exists* $(x, B) \in \mathbb{D}(R)$ *such that for every* $i \in N$, not$[O_i\ P_i^O\ B_i]$.

Proof: From the second theorem of welfare economics, the existence of $(x, B) \in \mathbb{D}(R)$ is guaranteed. Suppose to the contrary that, for some $i \in N$, $O_i\ P_i^O\ B_i$. Since $R_i^O \in \hat{\mathcal{R}}^O$, it must be true that there exists $y_i \in O_i$ with $y_i P_i^A z_i$ for every $z_i \in B_i$. In particular, $y_i P_i^A x_i$. Because (x, O) is autonomous, $x_i R_i^A w_i$ for every $w_i \in O_i$, and hence $x_i R_i^A y_i$. This is a contradiction. ■

Proposition 9 *Let* $R = (R_1, \ldots, R_n) \in \mathcal{R}^n$ *be such that for every* $i \in N, R_i^O \in \hat{\mathcal{R}}^O$ *and* R_i *satisfies Condition A. Let* $(x, O) \in \mathbb{Z}(R)$ *with* $x_i >> 0$ *for every* $i \in N$. *If* (x, O) *is allocation-Pareto optimal for* R*, then there exists* $(x, B) \in \mathbb{D}(R)$ *such that for every* $i \in N$, not$[(x_i, O_i)\bar{P}_i(x_i, B_i)]$.

Proof: Suppose, to the contrary, that for some $i \in N$, $(x_i, O_i)\bar{P}_i(x_i, B_i)$. Since R_i satisfies Condition A, and $x_i I_i^A x_i$, we must have $O_i P_i^O B_i$. Following a similar arguement as in the proof of Proposition 8, we reach a contradiction. ■

The final two propositions in this section state that for *any* configuration (x, O) (which may be opportunity Pareto optimal or overall Pareto optimal), we can find a decentralizable configuration (y, B) that is no worse than (x, O) for any agent with respect to his allocation preferences, opportunity preferences, and/or overall preferences.

Proposition 10 *Let $R = (R_1, \ldots, R_n) \in \mathcal{R}^n$ be such that for every $i \in N, R_i^O \in \hat{\mathcal{R}}^O$. Let $(x, O) \in \mathbb{Z}(R)$ with $x_i >> 0$ for every $i \in N$. Then, there exists $(y, B) \in \mathbb{D}(R)$ such that for every $i \in N$, $y_i \ R_i^A \ x_i$ and* not$[O_i \ P_i^O \ B_i]$.

Proof: Under our assumptions on R_i^A for each $i \in N$, and the assumption that $x_i >> 0$ for every $i \in N$, there exists a Walras equilibrium for the initial endowments $(x_1, \cdots, x_n)$, that is, there exist a price vector $p \in \mathbb{R}_+^k$ and an allocation $y \in \mathbb{R}_+^{nk}$ such that $(y, B) \in \mathbb{D}(R)$ with $B \equiv (B(p, x_1), \cdots, B(p, x_n)) = (B(p, y_1), \cdots, B(p, y_n))$. Then, for every $i \in N$, $y_i \ R_i^A \ x_i$. Following similar arguments as in the proof of Proposition 8, we can show that for every $i \in N$, not$[O_i P_i^O B_i]$. ■

Proposition 11 *Let $R = (R_1, \ldots, R_n) \in \mathcal{R}^n$ be such that for every $i \in N, R_i^O \in \hat{\mathcal{R}}^O$ and R_i satisfies Condition A. Let $(x, O) \in \mathbb{Z}(R)$ with $x_i >> 0$ for every $i \in N$. Then, there exists $(y, B) \in \mathbb{D}(R)$ such that for every $i \in N$, $y_i \ R_i^A \ x_i$,* not$[O_i \ P_i^O \ B_i]$ *and* not$[(x_i, O_i)\bar{P}_i(y_i, B_i)]$.

Proof: The proof is similar to that of Proposition 10, and it is omitted. ■

Recall our remark in Section 5 that the type of an agent's opportunity-preference quasiorders discussed by Sen indicates the agent comes very close to being a consequentialist. It is therefore easy to see that, in our framework, if all the agents' opportunity-preference quasi-orders are the type discussed by Sen, the classical second welfare theorem can be extended without much difficulty.

6 Opportunity Preferences with Superset Domination

So far we have seen that if opportunity preference quasiorders are restricted to the class $\hat{\mathcal{R}}^O$, the two welfare theorems can be extended without easily. However, the class $\hat{\mathcal{R}}^O$ of opportunity-preference quasiorders represents only one of possible types of preferences for opportunities. Let us next consider opportunity preference quasiorders satisfying the following "superset domination" condition. Let μ be the Lebesgue measure on $\mathbb{R}^k$.

Condition S: For all $A_i, B_i \in \mathcal{O}$,
(i) $A_i \supseteq B_i \Rightarrow A_i \; R_i^O \; B_i$, and
(ii) $A_i \supseteq B_i$ and $\mu(A_i) > \mu(B_i) \Rightarrow A_i \; P_i^O \; B_i$.

Let $\tilde{\mathcal{R}}^O$ be the class of opportnity preference quasiorders satisfying Condition S. Many opportunity preference quasiorders discussed in the literature on ranking opportunity sets belong to $\tilde{\mathcal{R}}^O$. On the other hand, notice that no opportunity-preference quasiorder in $\hat{\mathcal{R}}^O$ is in $\tilde{\mathcal{R}}^O$. In this section, we examine the possibility of extending the two welfare theorems when opportunity-preference quasiorders are elements of $\tilde{\mathcal{R}}^O$.

We consider first the opportunity preference quasiorder that ranks opportunity sets only by inclusion relations: for all $A_i, B_i \in \mathcal{O}$, $A_i \; R_i^O \; B_i$ if and only if $A_i \supseteq B_i$. With this opportunity preference quasiorder, however, a decentralizable configuration is not necessarily opportunity-Pareto-optimal as the following example shows.

Example 1 Let $N = \{1, 2\}$, $k = 2$ and $R = (R_1, R_2)$. For every $i \in N$, and for all $A_i, B_i \in \mathcal{O}$, $A_i \; R_i^O \; B_i$ if and only if $B_i \subseteq A_i$. For every $i \in N$, R_i^A is represented by the following utility function:

$$u_i(x_{i1}, x_{i2}) = \begin{cases} 5x_{i2} & \text{if} \quad x_{i2} \leq \frac{1}{2}x_{i1} \\ 2x_{i1} + x_{i2} & \text{if} \quad \frac{1}{2}x_{i1} < x_{i2} < 2x_{i1} \\ 4x_{i1} & \text{if} \quad 2x_{i1} \leq x_{i2} \end{cases}$$

Let $\omega = (10, 10)$. Define $(x^*, B) \in \mathbb{Z}$ as follows: for each $i \in N$, $x_i^* \equiv (5, 5)$ and $B_i \equiv B(p, x_i^*)$ where $p \equiv (2, 1)$. Then, $(x^*, B) \in \mathbb{D}(R)$. For each $i \in N$, define

$$C_i \equiv B_i \cup \{x_i \in \mathbb{R}_+^2 \mid 0 \leq x_{i1} \leq 10, \; x_{i2} = 0\}.$$

Clearly, for every $i \in N$, $C_i \supseteq B_i$ and $C_i \neq B_i$, and hence $C_i \ P_i \ B_i$. Moreover, (x^*, C) is autonomous for R. Thus, (x^*, B) is not opportunity-Pareto-optimal for R. ■

Next, we consider another interesting subclass of $\tilde{\mathcal{R}}^O$, namely the class of "additive" opportunity preference orders. Let $i \in N$ and $R_i^O \in \mathcal{R}^O$. We say that R_i^O is *additive* if there exists an integrable function $f_i : \mathbb{R}_+^k \to \mathbb{R}_{++}$ such that for all $A_i, B_i \in \mathcal{O}$,

$$A_i \ R_i^O \ B_i \Leftrightarrow \int_{A_i} f_i d\mu \geq \int_{B_i} f_i d\mu.$$

Clearly, if R_i^O is additive, then $R_i^O \in \tilde{\mathcal{R}}^O$. A case of particular interest is when the function f_i coincides with the utility function $u_i : \mathbb{R}_+^k \to \mathbb{R}_{++}$ representing the allocation preference order. In this case, opportunity sets are ranked according to the sum of the utilities of possible consumption bundles in each set.

The following example shows that when every agent has an additive opportunity preference order, a decentralizable configuration is not necessarily opportunity-Pareto-optimal. Moreover, there exists a Pareto-optimal allocation $x^* >> 0$ such that no decentralizable configuration that supports x^* (that is, $(x, B) \in \mathbb{D}(R)$ with $x = x^*$) is opportunity-Pareto-optimal.[7] Thus, neither the first nor the second fundamental welfare theorem cannnot be extended to this class of opportunity preference orders.

Example 2 Let $N = \{1, 2\}$, $k = 2$ and $R \equiv (R_1, R_2)$. Each $i \in N$ has the following utility function $u_i : \mathbb{R}_+^2 \to \mathbb{R}$ defined by:

$$u_i(x_{i1}, x_{i2}) = \begin{cases} 5x_{i2} & \text{if} \quad x_{i2} \leq \frac{1}{4}x_{i1} \\ x_{i1} + x_{i2} & \text{if} \quad \frac{1}{4}x_{i1} < x_{i2} < \frac{3}{2}x_{i1} \\ \frac{5}{2}x_{i1} & \text{if} \quad \frac{3}{2}x_{i1} \leq x_{i2} \end{cases}$$

For each $i \in N$, and for all $A_i, B_i \in \mathcal{O}$,

$$A_i \ R_i^O \ B_i \Leftrightarrow \int_{A_i} u_i d\mu \geq \int_{B_i} u_i d\mu.$$

[7] Note, however, that there always exists a decentralizable configuration that is opportunity-Pareto-optimal. Simply give one agent all resources, and let his opportunity set be the whole Edgeworth box.

Let $\omega \equiv (10, 10)$. Define $(x^*, B) \in \mathbb{Z}$ as follows: for each $i \in N$, $x_i^* \equiv (5, 5)$ and $B_i \equiv B(p^*, x_i^*)$ where $p^* \equiv (1, 1)$. Then, $(x^*, B) \in \mathbb{D}(R)$. Moreover, (x^*, B) is the unique decentralizable configuration that supports the Pareto-optimal allocation x^*. For each $i \in N$, define

$$\begin{aligned} C_i &\equiv \{x_i \in \mathbb{R}_+^2 \mid x_i \in B_i \text{ and } x_{i2} \leq 9\} \\ &\quad \cup \{x_i \in \mathbb{R}_+^2 \mid 0 \leq x_{i2} \leq 1 \text{ and } 0 \leq x_{i1} \leq 10\}. \end{aligned}$$

Clearly, (x^*, C) is autonomous for R. It can be calculated that for every $i \in N$,

$$\int_{C_i} u_i d\mu > \int_{B_i} u_i d\mu.$$

Hence, for every $i \in N$, $C_i\ P_i^O\ B_i$. Thus, (x^*, B) is not opportunity-Pareto-optimal. ■

7 Concluding Remarks

To some limited extent, we can extend the fundamental welfare theorems to incorporate not only allocation Pareto optimality but opportunity Pareto optimality and overall Pareto optimality as well. When agents' opportunity-preferences are of the types discussed in Sections 4 and 5, the market mechanism may be regarded as appealing since it generates configurations that are allocation Pareto optimal, opportunity Pareto optimal and overall Pareto optimal (extensions of the first fundamental theorem of welfare economics). Furthermore, for every configuration that is allocation Pareto optimal, we can find a market mechanism to support it, and for every configuration that is either opportunity Pareto optimal or overall Pareto optimal, we can find a market mechanism to achieve a configuration that is not worse than the given configuration in terms of allocation preferences, opportunity preferences and overall preferences (extensions of the second fundamental theorem of welfare economics).

As we have also shown, however, that the above results depend crucially on the particular classes of opportunity preference relations. If we go beyond these classes, we may encounter difficulties in extending the two theorems of welfare economics. The difficulties point to the incompatibility of allocation Pareto optimality, opportunity Pareto optimality, and/or overall Pareto optimality. We have used some specific class of opportunity preference relations to illustrate the possible conflict. Investigation into various types of opportunity-preference relations may deserve further exploration.

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