Informational Requirements for Social Choice in Economic Environments*

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Abstract

Arrow's celebrated theorem shows that the aggregation of individuals' preferences into a social ordering cannot make the ranking of any pair of alternatives depend only on individuals' preferences over that pair, unless the fundamental Pareto and non-dictatorship principles are violated. In a unified approach covering the theory of social choice and the theory of fair allocation, we investigate how much information is needed to rank a pair of allocations by social ordering functions and by allocation rules satisfying the Pareto principle and anonymity. In the standard model of division of commodities, we show that knowledge of a good portion of indifference hypersurfaces is needed for social ordering functions, whereas allocation rules require only knowledge of marginal rates of substitution.

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1 Introduction

From Arrow's celebrated theorem of social choice, it is well known that the aggregation of individuals' preferences into a social ordering cannot make the social ranking of any pair of alternatives depend only on individuals' preferences over that pair (this is the famous axiom of Independence of Irrelevant Alternatives). Or, more precisely, it cannot do so without trespassing basic requirements of unanimity (the Pareto principle) and anonymity (even the very weak condition of non-dictatorship). This raises the following question: What additional information about preferences would be needed in pairwise comparisons in order to make the aggregation of preferences possible, and compatible with the basic requirements of unanimity and anonymity?

In the last decades, the literature on social choice has explored several paths and gave interesting answers to this question. The main avenue of research has been, after Sen (1970) and d'Aspremont and Gevers (1977), the introduction of information about utilities, and it has been shown that the classical social welfare functions, and less classical ones, could be obtained with the Arrovian axiomatic method by letting the social preferences take account of specific kinds of utility information. Another, very important, approach, initiated by Foley (1967), Kolm (1972) and Varian (1974) among others,[1] has studied the problem of fair allocation in economic models and has managed to get round Arrow–like impossibility and to propose a myriad of nicely behaved allocation rules. Although the usual interpretation of this success rests on the fact that allocation rules do not aggregate preferences but only selects a subset of first-best allocations, the relationships between the theory of fair allocation and the theory of social choice have remained rather loose up to now.

In this paper, we focus on the introduction of additional information about preferences that is not of the utility sort. In other words, we retain a framework with purely ordinal, non-comparable preferences. The kind of information that we study is about the shape of indifference curves, and we ask how much one needs to know about the indifference curves to rank a pair of allocations by social ordering functions satisfying the unanimity and anonymity requirements. The introduction of this additional information is formulated here in terms of weakening Arrow's axiom of independence of irrelevant alternatives. As shown below, it turns out that a good deal of

[1] For a survey, see Moulin and Thomson (1997).

information is needed. Purely local information like marginal rates of substitution would not do, and we establish an extension of Arrow's theorem to this kind of information.

We also study the same question about allocation rules, whose particular feature is that they dichotomize all allocations between the desirable ones and the rest. We show that additional information is also needed there, although in a less demanding way. For instance, information about marginal rates of substitution is enough to get out of Arrow-like impossibility.

The framework adopted here is an economic model, namely, the canonical model of division of infinitely divisible commodities among a finite set of agents. We chose to study an economic model rather than the abstract model that is now commonly used in the theory of social choice[2] for three reasons. First, it allows a more fine-grained analysis of the information about preferences, because it makes it sensible to talk about marginal rates of substitution and other local notions about indifference curves. Second, in an economic model preferences are naturally restricted, and by considering a restricted domain we can hope to obtain positive results with less information than under unrestricted domain. Third, it makes it possible to compare the informational requirements for social ordering functions and allocation rules in a context that is relevant to the existing literature on fair allocation. In this way, we are able to contribute to bridging the gap between the theory of social choice and the theory of fair allocation.

The motivation for our research draws on many straws taken from recent and less recent literature. Attempts to construct social ordering functions and similar objects embodying unanimity and equity requirements were made by Suzumura (1981a,b, 1983) and Tadenuma (1998). Fleurbaey (1996) and Roemer (1996) noticed that most allocation rules in the theory of fair allocation violate conditions akin to the Arrow independence of irrelevant alternatives, although they do not use utility information. The idea that information about indifference curves is sufficient, hinted at by Pazner and Schmeidler (1978) and Maniquet (1994), was revived by Bossert, Fleurbaey and Van de gaer (1999) and Fleurbaey and Maniquet (1996, 2000) who were able to construct nicely behaved social ordering functions on this basis. Campbell and Kelly (2000) recently studied essentially the same issue in the abstract model of social choice, and showed that limited information about preferences may

[2] Recollect, however, that Arrow's initial presentations (1950, 1951) dealt with this economic model of division of commodities.

be enough, although they focus on non-dictatorship and do not study how much information is necessary with the stronger requirement of anonymity. Le Breton (1997), in a nice synthesis, presented a unified view of the theory of social choice and the theory of fair allocation, but without emphasizing the issue of the informational basis.

The paper is organized as follows. The next section introduces the framework and the main notions. The results are presented in Section 3, for social ordering functions, and in Section 4, for allocation rules. Section 5 concludes. The appendix contains some proofs.

2 Model and Definitions

2.1 The Model

The *population* is fixed. Let $N = \{1, ..., n\}$ be the set of *agents* where $2 \leq n < \infty$. There are ℓ *goods* indexed by $k = 1, ..., \ell$ where $2 \leq \ell < \infty$. Agent i's *consumption bundle* is a vector $x_i = (x_{i1}, ..., x_{i\ell})$. An *allocation* is a vector $x = (x_1, ..., x_n)$. The *set of allocations* is $\mathsf{R}_+^{n\ell}$. The set of allocations such that no individual bundle x_i is equal to the zero vector is denoted X.

A *preordering* is a reflexive and transitive binary relation. Agent i's *preferences* are described by a complete preordering R_i (strict preference P_i, indifference I_i) on R_+^ℓ. A *profile of preferences* is denoted $\mathsf{R} = (R_1, ..., R_n)$. Let $\mathcal{R}$ be the set of continuous, convex, and strictly monotonic preferences over R_+^ℓ.

There is no production,[3] and the amount of *total resources* is a given $\omega \in \mathsf{R}_{++}^\ell$. All allocations x such that $\sum_{i \in N} x_i \leq \omega$ are said to be *feasible.*[4] Let

$$F = \{x \in \mathsf{R}_+^{n\ell} | \sum_{i \in N} x_i = \omega\}.$$

Notice that all our results would remain true under the assumption of free disposal, that is, under the alternative definition of F as $\{x \in \mathsf{R}_+^{n\ell} | \sum_{i \in N} x_i \leq \omega\}$. Let $E(\mathsf{R})$ denote the set of *Pareto-efficient allocations.* Because of strict monotonicity of preferences, there is no need to distinguish Pareto-efficiency in the strong sense and in the weak sense.

[3]Our results about social ordering functions could be extended with little change to the case when production is possible.

[4]Vector inequalities are denoted as usual: $\geq, >$, and $\gg$.

A *social ordering function* (SOF) is a mapping $\bar{R}$ defined on $\mathcal{R}^n$, such that for all $\mathsf{R} \in \mathcal{R}^n$, $\bar{R}(\mathsf{R})$ is a complete preordering on the set of allocations $\mathsf{R}_+^{n\ell}$. Let $\bar{P}(\mathsf{R})$ (resp. $\bar{I}(\mathsf{R})$) denote the related strict preference (resp. indifference) relations.

An *allocation rule* (AR) is a set-valued mapping S defined on $\mathcal{R}^n$, such that for all $\mathsf{R} \in \mathcal{R}^n$, $S(\mathsf{R})$ is a non-empty subset of F. An AR is *essentially single-valued* if all selected allocations are Pareto-indifferent:

$$\forall x, y \in S(\mathsf{R}), \forall i \in N, \; x_i I_i y_i.$$

An alternative definition of SOFs and ARs makes them a function of ω as well as R. This is useful when changes in ω are studied, but here we focus only on the information about preferences, and since ω is kept fixed throughout the paper, we omit this argument.

An AR dichotomizes the set of all allocations between the desirable ones and the rest. Hence, it can be regarded as a "two-tier" SOF. The fact that ARs are just a particular kind of SOF allows us to study the informational bases for SOFs and ARs in a unified way. In particular, the axioms of independence which, as presented below, express the informational requirements for SOFs, can be directly applied to ARs, without restriction.

Let π be a bijection on N. For any $x \in \mathsf{R}_+^{n\ell}$, define $\pi(x) = (x'_1, ..., x'_n) \in \mathsf{R}_+^{n\ell}$ by $x'_i = x_{\pi(i)}$ for all $i \in N$, and for any $\mathsf{R} \in \mathcal{R}^n$, define $\pi(\mathsf{R}) = (R'_1, ..., R'_n) \in \mathcal{R}^n$ by $R'_i = R_{\pi(i)}$ for all $i \in N$. Let Π be the set of all bijections on N. The basic requirements of unanimity and anonymity on which we focus in this paper are the following.

Weak Pareto for SOF: $\forall x, y \in \mathsf{R}_+^{n\ell}, \forall \mathsf{R} \in \mathcal{R}^n$ if $\forall i \in N, \; x_i P_i y_i$, then $x \bar{P}(\mathsf{R}) y$.

This axiom cannot be applied to ARs, because it requires a too fine-grained ranking of allocations. The usual practice in the theory of fair allocation is to require the selected allocations to be Pareto-efficient.

Pareto for AR: $\forall \mathsf{R} \in \mathcal{R}^n$, $S(\mathsf{R}) \subset E(\mathsf{R})$.

Anonymity for SOF: $\forall x, y \in \mathsf{R}_+^{n\ell}$, $\forall \mathsf{R} \in \mathcal{R}^n$, $\forall \pi \in \Pi$,

$$x \bar{R}(\mathsf{R}) y \Leftrightarrow \pi(x) \, \bar{R}(\pi(\mathsf{R})) \, \pi(y).$$

This axiom may be directly applied to ARs, although it is worthwhile to notice that it then boils down to the following simple condition.

Anonymity for AR: $\forall \mathsf{R} \in \mathcal{R}^n$, $\forall \pi \in \Pi$, $\forall x \in S(\mathsf{R})$, $\pi(x) \in S(\pi(\mathsf{R}))$.

Concerning the non-dictatorship form of anonymity, we only define here what dictatorship means, for convenience. Notice that it has to do only with allocations in X, that is, without any zero bundle.

Dictatorial SOF: The SOF $\bar{R}$ is dictatorial if there exists $i_0 \in N$ such that:

$$\forall x, y \in X, \forall \mathsf{R} \in \mathcal{R}^n, x_{i_0} P_{i_0} y_{i_0} \Rightarrow x\bar{P}(\mathsf{R})y.$$

Again, this definition is not meaningful for ARs, since it cannot be observed among "two-tier" rankings. Following Le Breton (1997), and in view of monotonicity of individual preferences, we propose the following adapted definition for ARs.

Dictatorial AR: The AR S is dictatorial if there exists $i_0 \in N$ such that:

$$\forall \mathsf{R} \in \mathcal{R}^n,\ S(\mathsf{R}) = \{x \in \mathsf{R}_+^{nl} | x_{i_0} = \omega\}.$$

It is also worth introducing the following axiom, which is specific to ARs.[5]

Equal Treatment of Equals (for AR): $\forall \mathsf{R} \in \mathcal{R}^n$, $\forall x \in S(\mathsf{R})$, $\forall i, j \in N$, if $R_i = R_j$, then $x_i I_i x_j$.

Lemma 1 *Any essentially single-valued AR satisfying Anonymity also satisfies Equal Treatment of Equals.*

2.2 Variants of Independence of Irrelevant Alternatives

The traditional, Arrovian, version of Independence of Irrelevant Alternatives is:

Independence of Irrelevant Alternatives (IIA): $\forall x, y \in \mathsf{R}_+^{n\ell}$, $\forall \mathsf{R}, \mathsf{R}' \in \mathcal{R}^n$, if $\forall i \in N,\ x_i R_i y_i \Leftrightarrow x_i R_i' y_i$, then $x\bar{R}(\mathsf{R})y \Leftrightarrow x\bar{R}(\mathsf{R}')y$.

Notice that it would be equivalent to write the conclusion as:

$$x\bar{P}(\mathsf{R})y \Leftrightarrow x\bar{P}(\mathsf{R}')y.$$

This remark will be useful when adapting this condition to ARs.

[5]Related conditions can be defined for SOFs. See footnote 8.

It is possible to weaken IIA by strengthening the premise. This amounts to allowing the SOF to make use of more information when ranking any pair of allocations.

We first consider the possibility for the SOF to take account of marginal rates of substitution. Economists are used to focus on marginal rates of substitution when assessing the efficiency of an allocation, especially under convexity, since for convex preferences the marginal rates of substitution determine the half space in which the upper contour set lies. Moreover, for efficient allocations, shadow prices enable one to compute the relative implicit income shares of different agents, thereby potentially providing a relevant measure of inequalities in the distribution of resources. Therefore, taking account of marginal rates of substitution is a natural extension of the informational basis of social choice in economic environments. Let $C(x_i, R_i)$ denote the cone of price vectors that support the upper contour set for R_i at x_i :

$$C(x_i, R_i) = \{p \in \mathsf{R}^\ell | \forall y \in \mathsf{R}^\ell_+, py = px_i \Rightarrow x_i R_i y\}.$$

When preferences R_i are strictly monotonic, one has $C(x_i, R_i) \subset \mathsf{R}^\ell_{++}$ whenever $x_i \gg 0$.

IIA with Marginal Rates of Substitution (IIA-MRS): $\forall x, y \in \mathsf{R}^{n\ell}_+$, $\forall \mathsf{R}, \mathsf{R}' \in \mathcal{R}^n$, if $\forall i \in N$, $x_i R_i y_i \Leftrightarrow x_i R'_i y_i$ and $C(x_i, R_i) = C(x_i, R'_i)$, $C(y_i, R_i) = C(y_i, R'_i)$, then $x\bar{R}(\mathsf{R})y \Leftrightarrow x\bar{R}(\mathsf{R}')y$.

Marginal rates of substitution give an infinitesimally local piece of information about preferences at given allocations. It would be interesting to take account of the preferences over some finitely sized neighborhoods of the two allocations. Define, for any small real number $\varepsilon > 0$,

$$B_\varepsilon(x_i) = \{v \in \mathsf{R}^\ell_+ | \max_{k \in \{1,\ldots,\ell\}} |x_{ik} - v_k| \leq \varepsilon\}$$

IIA with ε-Neighborhoods (IIA-εN): $\forall x, y \in \mathsf{R}^{n\ell}_+$, $\forall \mathsf{R}, \mathsf{R}' \in \mathcal{R}^n$, if $\forall i \in N$, $x_i R_i y_i \Leftrightarrow x_i R'_i y_i$ and $\forall (z, z') \in B_\varepsilon(x_i)^2 \cup B_\varepsilon(y_i)^2, z R_i z' \Leftrightarrow z R'_i z'$, then $x\bar{R}(\mathsf{R})y \Leftrightarrow x\bar{R}(\mathsf{R}')y$.

An alternative extension of the informational basis allows the SOF to take account of parts of indifference hypersurfaces. The *indifference sets* are defined as

$$I(x_i, R_i) = \{z \in \mathsf{R}^\ell_+ \mid z I_i x_i\}.$$

Here we consider two ways of focusing on parts of such sets. First, it is natural to focus on the part of indifference sets which lies within the feasible set. However, when considering any pair of allocations, the two allocations may need different amounts of total resources to be feasible. Therefore we need to introduce the following notions. The smallest amount of total resources which makes two allocations x and y feasible can be defined by $\omega(x,y) = (\omega_1(x,y),...,\omega_\ell(x,y))$, where $\omega_k(x,y) = \max\{\sum_{i\in N} x_{ik}, \sum_{i\in N} y_{ik}\}$ for all $k \in \{1,...,\ell\}$. For any vector $t \in \mathsf{R}^\ell_+$, define the set $\Omega(t) \subset \mathsf{R}^\ell_+$ by

$$\Omega(t) = \left\{z \in \mathsf{R}^\ell_+ \mid z \leq t\right\}$$

The following axiom captures the idea that the ranking of two allocations should depend only on the indifference sets, and on preferences over the minimal subset in which the two allocations are feasible.

IIA with Indifference Sets on Feasible Allocations (IIA-ISFA): $\forall x, y \in \mathsf{R}^{n\ell}_+$, $\forall \mathsf{R}, \mathsf{R}' \in \mathcal{R}^n$, if $\forall i \in N, I(x_i, R_i) \cap \Omega(\omega(x,y)) = I(x_i, R'_i) \cap \Omega(\omega(x,y))$, $I(y_i, R_i) \cap \Omega(\omega(x,y)) = I(y_i, R'_i) \cap \Omega(\omega(x,y))$, then $x\bar{R}(\mathsf{R})y \Leftrightarrow x\bar{R}(\mathsf{R}')y$.

It will actually be worth considering a much weaker axiom, which relies on radial expansions of the minimal feasible set in which the two allocations to be compared are feasible. A radial expansion is defined as follows: for any set $Q \subset \mathsf{R}^\ell$ and any $\lambda \geq 1$,

$$\lambda Q = \{q \in \mathsf{R}^\ell | \lambda^{-1} q \in Q\}.$$

The next axiom is very weak since it allows the radial factor λ to be arbitrarily large.

IIA with Indifference Sets on Expanded Feasible Allocations (IIA-ISEFA): $\exists \lambda \geq 1, \forall x, y \in \mathsf{R}^{n\ell}_+$, $\forall \mathsf{R}, \mathsf{R}' \in \mathcal{R}^n$, if $\forall i \in N, I(x_i, R_i) \cap \lambda\Omega(\omega(x,y)) = I(x_i, R'_i) \cap \lambda\Omega(\omega(x,y))$, $I(y_i, R_i) \cap \lambda\Omega(\omega(x,y)) = I(y_i, R'_i) \cap \lambda\Omega(\omega(x,y))$, then $x\bar{R}(\mathsf{R})y \Leftrightarrow x\bar{R}(\mathsf{R}')y$.

A second way of extending the information about indifference sets is to rely on a path

$$\Lambda_{\omega_0} = \{\lambda\omega_0 \in \mathsf{R}^\ell_{++} | \lambda \in \mathsf{R}_+\},$$

where $\omega_0 \in \mathsf{R}^\ell_{++}$ is fixed, and to focus on the part of the indifference sets which belongs to this path. The idea of referring to such a path has been introduced by Pazner and Schmeidler (1978), and may be justified if the path contains relevant benchmark bundles. The choice of ω_0 is not discussed here, but it need not be arbitrary. For instance, one may imagine that it could rely on appropriate equity conditions.

IIA with Indifference Sets on Path ω_0 (IIA-ISPω_0): $\forall x, y \in \mathsf{R}_+^{n\ell}$, $\forall \mathsf{R}, \mathsf{R}' \in \mathcal{R}^n$, if $\forall i \in N, I(x_i, R_i) \cap \Lambda_{\omega_0} = I(x_i, R_i') \cap \Lambda_{\omega_0}$, $I(y_i, R_i) \cap \Lambda_{\omega_0} = I(y_i, R_i') \cap \Lambda_{\omega_0}$, then $x\bar{R}(\mathsf{R})y \Leftrightarrow x\bar{R}(\mathsf{R}')y$.

The last extension of informational basis that we consider is to introduce whole indifference hypersurfaces. This condition was already introduced and studied by Hansson (1973) in the abstract model of social choice, who showed that the Borda rule satisfies it.

IIA with Whole Indifference Sets (IIA-WIS): $\forall x, y \in \mathsf{R}_+^{n\ell}$, $\forall \mathsf{R}, \mathsf{R}' \in \mathcal{R}^n$, if $\forall i \in N,\ I(x_i, R_i) = I(x_i, R_i'),\ I(y_i, R_i) = I(y_i, R_i')$, then $x\bar{R}(\mathsf{R})y \Leftrightarrow x\bar{R}(\mathsf{R}')y$.

Lemma 2 *For all $\varepsilon > 0$,*

$$\begin{array}{ccc} & \textit{IIA-}\varepsilon N & \\ & \Uparrow & \\ \textit{IIA} \Longrightarrow & \left\{ \begin{array}{c} \textit{IIA-MRS} \\ \textit{IIA-ISFA} \Rightarrow \textit{IIA-ISEFA} \\ \textit{IIA-ISP}\omega_0 \end{array} \right\} & \Rightarrow \textit{IIA-WIS}. \end{array}$$

2.3 Independence of Irrelevant Alternatives for Allocation Rules

IIA is an axiom for SOFs. It is not obvious how one can translate this for ARs. One way is to consider ARs as two-tier SOFs. Recall that the conclusion of IIA axioms for SOFs can be written

$$x\bar{P}(\mathsf{R})y \Leftrightarrow x\bar{P}(\mathsf{R}')y.$$

In the case of ARs, $x\bar{P}(\mathsf{R})y$ reads:

$$x \in S(\mathsf{R}) \text{ and } y \notin S(\mathsf{R}).$$

This suggests a direct translation of the above IIA family of axioms.[6]

[6] We actually make a slight change, by applying the IIA axioms for ARs to allocations in F only. This makes the axioms slightly weaker, and we chose to do this because it is interesting to check that our results do not depend on considering infeasible allocations.

Independence of Irrelevant Alternatives (IIA): $\forall x, y \in F$, $\forall \mathsf{R}, \mathsf{R}' \in \mathcal{R}^n$, if $\forall i \in N,\ x_i R_i y_i \Leftrightarrow x_i R'_i y_i$, then $[x \in S(\mathsf{R})$ and $y \notin S(\mathsf{R})] \Leftrightarrow [x \in S(\mathsf{R}')$ and $y \notin S(\mathsf{R}')]$.

IIA with Marginal Rates of Substitution (IIA-MRS): $\forall x, y \in F$, $\forall \mathsf{R}, \mathsf{R}' \in \mathcal{R}^n$, if $\forall i \in N,\ x_i R_i y_i \Leftrightarrow x_i R'_i y_i$ and $C(x_i, R_i) = C(x_i, R'_i)$, $C(y_i, R_i) = C(y_i, R'_i)$, then $[x \in S(\mathsf{R})$ and $y \notin S(\mathsf{R})] \Leftrightarrow [x \in S(\mathsf{R}')$ and $y \notin S(\mathsf{R}')]$.

IIA with Indifference Sets on Feasible Allocations (IIA-ISFA): $\forall x, y \in F$, $\forall \mathsf{R}, \mathsf{R}' \in \mathcal{R}^n$, if $\forall i \in N, I(x_i, R_i) \cap \Omega(\omega) = I(x_i, R'_i) \cap \Omega(\omega)$, $I(y_i, R_i) \cap \Omega(\omega) = I(y_i, R'_i) \cap \Omega(\omega)$, then $[x \in S(\mathsf{R})$ and $y \notin S(\mathsf{R})]$ $\Leftrightarrow [x \in S(\mathsf{R}')$ and $y \notin S(\mathsf{R}')]$.

IIA with Whole Indifference Sets (IIA-WIS): $\forall x, y \in F$, $\forall \mathsf{R}, \mathsf{R}' \in \mathcal{R}^n$, if $\forall i \in N,\ I(x_i, R_i) = I(x_i, R'_i)$, $I(y_i, R_i) = I(y_i, R'_i)$, then $[x \in S(\mathsf{R})$ and $y \notin S(\mathsf{R})] \Leftrightarrow [x \in S(\mathsf{R}')$ and $y \notin S(\mathsf{R}')]$.

Notice that an axiom based on the path ω_0 would not make sense here because the condition

$$I(x_i, R_i) \cap \Lambda_{\omega_0} = I(x_i, R'_i) \cap \Lambda_{\omega_0}$$

would not guarantee that x remains Pareto-efficient.

Consider the AR $S_{\bar{R}}$ related to a SOF $\bar{R}$ in the following way: for all $\mathsf{R} \in \mathcal{R}^n$,

$$S_{\bar{R}}(\mathsf{R}) = \{x \in F \mid \forall y \in Z,\ x\bar{R}(\mathsf{R})y\}.$$

It is worth noticing that even if $\bar{R}$ satisfies IIA for SOFs, and $S_{\bar{R}}$ is well-defined (that is, $S_{\bar{R}}(\mathsf{R}) \neq \emptyset$ for all $\mathsf{R} \in \mathcal{R}^n$), $S_{\bar{R}}$ need not satisfy IIA for ARs, although it must satisfy the following condition whenever the premiss of IIA holds:

$$x \in S_{\bar{R}}(\mathsf{R}) \text{ and } y \notin S_{\bar{R}}(\mathsf{R}) \Rightarrow y \notin S_{\bar{R}}(\mathsf{R}').$$

For instance, fix two allocations $x^*, y^* \in F$ and let $\bar{R}$ be defined by: for all $\mathsf{R} \in \mathcal{R}^n$,

$$x^* \bar{R}(\mathsf{R}) y^* \Leftrightarrow x_1^* R_1 y_1^*$$

and for all $x, y \in \mathsf{R}_+^{n\ell} \setminus \{x^*, y^*\}$, $x\bar{I}(\mathsf{R})y$ and $x^*\bar{P}(\mathsf{R})x$, $y^*\bar{P}(\mathsf{R})x$. One can see that $\bar{R}$ satisfies IIA for SOFs, but $S_{\bar{R}}$ does not satisfy IIA for ARs.

The same fact is true about all weaker IIA axioms.

Another way to look at independence conditions for ARs is to imagine independence conditions that bear only on one allocation. That would yield the following family of axioms. The first one is very strong, but it seems that the usual premise in IIA does not put any local constraint on x, and it will be shown later on that it is actually equivalent to IIA.

Independence of Preferences (IP): $\forall x \in \mathsf{R}_+^{n\ell}$, $\forall \mathsf{R}, \mathsf{R}' \in \mathcal{R}^n$, if $x \in S(\mathsf{R})$, then $x \in S(\mathsf{R}')$.

The next one, dealing with marginal rates of substitution, is essentially Nagahisa's (1991) 'Local Independence':[7]

Independence of Preferences except MRS (IP-MRS): $\forall x \in \mathsf{R}_+^{n\ell}$, $\forall \mathsf{R}, \mathsf{R}' \in \mathcal{R}^n$, if $x \in S(\mathsf{R})$ and for all $i \in N$, $C(x_i, R_i) = C(x_i, R_i')$, then $x \in S(\mathsf{R}')$.

The next axiom says that only the part of indifference sets concerning feasible allocations should matter.

Independence of Preferences except Indifference Sets on Feasible Allocations (IP-ISFA): $\forall x \in \mathsf{R}_+^{n\ell}$, $\forall \mathsf{R}, \mathsf{R}' \in \mathcal{R}^n$, if $x \in S(\mathsf{R})$ and for all $i \in N$, $I(x_i, R_i) \cap \Omega(\omega) = I(x_i, R_i') \cap \Omega(\omega)$, then $x \in S(\mathsf{R}')$.

Notice that this axiom is stronger than an axiom suggested by Le Breton (1997),[8] which states that only preferences over feasible allocations should matter:

Independence of Preferences except Feasible Bundles (IP-FB): $\forall \mathsf{R}, \mathsf{R}' \in \mathcal{R}^n$, if $\forall x, y \in F, \forall i \in N, x_i R_i y_i \Leftrightarrow x_i R_i' y_i$, then $S(\mathsf{R}) = S(\mathsf{R}')$.

[7] See also Yoshihara (1998).

[8] Le Breton's synthesis of the theory of social choice and the theory of fair allocation (Le Breton, 1997) is symmetrical to ours, and is based on the idea that a SOF can be viewed as a particular kind of AR (choice correspondence) defined over a rich set of agendas (an agenda is a subset of alternatives from which the AR makes selections), typically the set of all finite subsets of allocations. In his synthesis, the theory of fair allocation is characterized by the facts that the preferences domain of ARs is restricted, and that the agendas domain is restricted as well to have some specific structures (for instance, the Edgeworth box). The axiom IP-FB embodies the choice theoretical version of Arrow independence in this framework. One can see that under this independence condition, a larger agenda allows ARs to use more information about preferences when deciding the set of selected allocations. However, that framework seems less convenient than ours in order to address the question of what information be retained when comparing a given pair of allocations.

The last axiom is due to Maniquet (1994).

Independence of Preferences except Whole Indifference Sets (IP-WIS): $\forall x \in \mathsf{R}_+^{n\ell}$, $\forall \mathsf{R}, \mathsf{R}' \in \mathcal{R}^n$, if $x \in S(\mathsf{R})$ and for all $i \in N$, $I(x_i, R_i) = I(x_i, R_i')$, then $x \in S(\mathsf{R}')$.

Although these independence of preferences axioms may seem extremely restrictive, they are actually not really stronger than the previous IIA axioms.

Lemma 3 *IP⇔IIA. On the class of ARs that never select $x \in F$ with $x_i = \omega$ for some i, or are essentially single-valued, IP-MRS⇔IIA-MRS.*

Proof. IP⇔IIA. It is obvious that IP⇒IIA. For the converse, choose any i_0 and define x^0 by $x^0_{i_0} = \omega$ (and $x^0_i = 0$ for all $i \neq i_0$). If for all R one has $S(\mathsf{R}) = F$ then IP is satisfied. Suppose then that this is not the case, and let R be such that $S(\mathsf{R}) \neq F$.

First case: $x^0 \in S(\mathsf{R})$. Take any $y \notin S(\mathsf{R})$. By monotonicity of preferences, for all R',

$$\forall i \in N,\ y_i R_i x^0_i \Leftrightarrow y_i R_i' x^0_i.$$

Therefore $x^0 \in S(\mathsf{R}')$ and $y \notin S(\mathsf{R}')$. The latter implies $F \setminus S(\mathsf{R}) \subset F \setminus S(\mathsf{R}')$. Since $x^0 \in S(\mathsf{R}')$, one can show by a symmetrical argument that $F \setminus S(\mathsf{R}') \subset F \setminus S(\mathsf{R})$ implying $S(\mathsf{R}') = S(\mathsf{R})$.

Second case: $x^0 \notin S(\mathsf{R})$. Take any $x \in S(\mathsf{R})$. By monotonicity of preferences, for all R',

$$\forall i \in N, x_i R_i x^0_i \Leftrightarrow x_i R_i' x^0_i.$$

Therefore $x^0 \notin S(\mathsf{R})$ and $x \in S(\mathsf{R}')$, and more generally $S(\mathsf{R}) \subset S(\mathsf{R}')$. By a symmetrical argument based on $x^0 \notin S(\mathsf{R}')$, one shows that $S(\mathsf{R}') \subset S(\mathsf{R})$.

IP-MRS⇔IIA-MRS. It is obvious that IP-MRS⇒IIA-MRS. For the converse, let $x \in S(\mathsf{R})$ and R' be such that for all i, $C(x_i, R_i') = C(x_i, R_i)$.

First case: S never selects allocations where some agent has ω. Choose $y, z \in F$ such that $y_1 = z_2 = \omega$. By strict monotonicity of preferences, there exists $\mathsf{R}'' \in \mathcal{R}^n$ such that for all i, $C(x_i, R_i'') = C(x_i, R_i)$, $C(y_i, R_i'') = C(y_i, R_i)$, $C(z_i, R_i'') = C(z_i, R_i')$. Because $y_1 = \omega$, $y \notin S(\mathsf{R})$. By IIA-MRS, $x \in S(\mathsf{R}'')$ and $y \notin S(\mathsf{R}'')$. Because $z_2 = \omega$, $z \notin S(\mathsf{R}'')$. And by IIA-MRS again, $x \in S(\mathsf{R}')$ and $z \notin S(\mathsf{R}')$.

Second case: S is essentially single-valued. Choose $y, z \in F$ such that for all i, for all $a, b \in \{x_i, y_i, z_i\}$, $a \neq b$, either $a \gg b$ or $a \ll b$. By monotonicity of preferences, there exists $\mathsf{R}'' \in \mathcal{R}^n$ such that for all i, $C(x_i, R_i'') = C(x_i, R_i)$,

$C(y_i, R_i'') = C(y_i, R_i)$, $C(z_i, R_i'') = C(z_i, R_i')$. By essential single-valuedness of S and monotonicity of preferences, $y \notin S(\mathsf{R})$. By IIA-MRS, $x \in S(\mathsf{R}'')$ and $y \notin S(\mathsf{R}'')$. By essential single-valuedness again, $z \notin S(\mathsf{R}'')$. And by IIA-MRS again, $x \in S(\mathsf{R}')$ and $z \notin S(\mathsf{R}')$. ■

Notice that in the proof of the second equivalence, one only needs to find allocations which are not selected by the AR for two profiles with the same MRS at the allocations. We also have similar results for the ISFA and WIS versions of the axioms.

3 Social Ordering Functions Need Indifference Curves

Let us first recall the formulation of Arrow's theorem for this model (Bordes and Le Breton 1989).

Proposition 1 *If a SOF $\bar{R}$ satisfies Weak Pareto and IIA, then it is dictatorial.*

It turns out, unfortunately, that introducing information about marginal rates of substitution, in addition to pairwise preferences, does not make room for the existence of satisfactory SOFs. More formally, weakening IIA to IIA-MRS does not alter the dictatorship conclusion of Arrow's theorem.

Proposition 2 *If a SOF $\bar{R}$ satisfies Weak Pareto and IIA-MRS, then it is dictatorial.*

The proof of this Proposition is long and is relegated to the appendix.

Inada (1964, 1971) also considered marginal rates of substitution in an IIA-like axiom, but the difference from our work is that he looked for a local aggregator of preferences, namely a mapping defining a social marginal rate of substitution between goods and individuals, on the basis of individual marginal rates of substitution. The global SOF was then obtained by integrating the social marginal rates of substitution. Therefore his independence condition was somewhat weaker since the social preference over two given alternatives could depend on all marginal rates of substitution over paths going from the first alternative to the second one. On the other hand, we do not require differentiability of the social ordering, so that our result is not logically related to Inada's one.

The next proposition shows that as soon as one switches from IIA-MRS to IIA-εN, the dictatorship result is avoided, even for an arbitrarily small ε, although it remains impossible to achieve Anonymity, even for an arbitrarily large ε. Moreover, for a small ε, the non-dictatorial example given in the proof remains dictatorial for most allocations, and one can safely conjecture that any SOF which satisfies Weak Pareto and IIA-εN is largely dictatorial, because under IIA-εN there are many free triples.

Proposition 3 *Let $\varepsilon > 0$ be given. There exists a non-dictatorial SOF satisfying Weak Pareto and IIA-εN. However, there does not exist a SOF satisfying Weak Pareto, IIA-εN and Anonymity.*

Proof. For the possibility result, define $\bar{R}$ as follows: $x\bar{R}(\mathsf{R})y$ if either $x_1R_1y_1$ and $[\{z \in \mathsf{R}_+^\ell | x_1R_1z\} * B_\varepsilon(0)$ or $\{z \in \mathsf{R}_+^\ell | y_1R_1z\} * B_\varepsilon(0)]$, or $x_2R_2y_2$ and $[\{z \in \mathsf{R}_+^\ell | x_1R_1z\} \subseteq B_\varepsilon(0)$ and $\{z \in \mathsf{R}_+^\ell | y_1R_1z\} \subseteq B_\varepsilon(0)]$. For brevity, let $\Gamma(v)$ denote $\left[\{z \in \mathsf{R}_+^\ell | vR_1z\} \subseteq B_\varepsilon(0)\right]$. Weak Pareto and the absence of dictator are straightforwardly satisfied. IIA-εN is also satisfied because when $\Gamma(x_1)$ and $\Gamma(y_1)$ hold, one has $B_\varepsilon(0) \subseteq B_\varepsilon(x_1) \cap B_\varepsilon(y_1)$, and therefore $\Gamma(x_1)$ and $\Gamma(y_1)$ remain true if preferences are kept fixed on $B_\varepsilon(x_1)$ and $B_\varepsilon(y_1)$. It remains to check transitivity of $\bar{R}(\mathsf{R})$. First note the following property. If $\Gamma(v)$ and vR_1v', then $\Gamma(v')$. Assume that there exist $x, y, z \in \mathsf{R}_+^{n\ell}$ such that $x\bar{R}(\mathsf{R})y\bar{R}(\mathsf{R})z\bar{P}(\mathsf{R})x$. If $\Gamma(x_1)$ and $\Gamma(y_1)$ and $\Gamma(z_1)$, this is impossible because one should have $x_2R_2y_2R_2z_2P_2x_2$. If only one of the three conditions $\Gamma(x_1)$,$\Gamma(y_1)$,$\Gamma(z_1)$ is satisfied, it is similarly impossible because one should have $x_1R_1y_1R_1z_1P_1x_1$. Assume $\Gamma(x_1)$ and $\Gamma(y_1)$ hold, but not $\Gamma(z_1)$. Then $y\bar{R}(\mathsf{R})z\bar{P}(\mathsf{R})x$ requires $y_1R_1z_1P_1x_1$, which implies $\Gamma(z_1)$, a contradiction. Assume $\Gamma(x_1)$ and $\Gamma(z_1)$ hold, but not $\Gamma(y_1)$. Then $x\bar{R}(\mathsf{R})y\bar{R}(\mathsf{R})z$ requires $x_1R_1y_1R_1z_1$, which implies $\Gamma(y_1)$, a contradiction. Assume $\Gamma(y_1)$ and $\Gamma(z_1)$ hold, but not $\Gamma(x_1)$. Then $z\bar{P}(\mathsf{R})x\bar{R}(\mathsf{R})y$ requires $z_1P_1x_1R_1y_1$, which implies $\Gamma(x_1)$, a contradiction.

The proof of the impossibility is very similar to that of Proposition 4 and is omitted here. ■

With the introduction of non-local information about indifference curves, one is also able to avoid dictatorship, but incompatibility with Anonymity remains. The result is important because no SOF violating Anonymity will ever be considered acceptable.

Proposition 4 *There exists a non-dictatorial SOF satisfying Weak Pareto, IIA-ISFA. However, there does not exist a SOF satisfying Weak Pareto, IIA-ISEFA and Anonymity.*

The proof is in the appendix.

From Pazner and Schmeidler's (1978) contribution one can derive the following result, which shows that not much information is needed, although it must be substantially non-local.

Proposition 5 *There exists a SOF satisfying Weak Pareto, IIA-ISPω_0 and Anonymity.*

Proof. By continuity and monotonicity of preferences, the following utility functions

$$u_i(x_i) = \min\{\alpha \in \mathbb{R}_+ | \alpha\omega_0 R_i x_i\}$$

are well-defined and represent preferences R_i. Let $\bar{R}$ be defined by: $x\bar{R}(\mathbf{R})y$ whenever

$$\min\{u_i(x_i)|i \in N\} \geq \min\{u_i(y_i)|i \in N\}.$$

This SOF clearly satisfies Weak Pareto and Anonymity. It also satisfies IIA-ISPω_0 because when $I(x_i, R_i) \cap \Lambda_{\omega_0} = I(x_i, R_i') \cap \Lambda_{\omega_0}$, one has

$$\min\{\alpha \in \mathbb{R}_+ | \alpha\omega_0 R_i x_i\} = \min\{\alpha \in \mathbb{R}_+ | \alpha\omega_0 R_i' x_i\}.$$

■

Notice that one could have the Strong Pareto property[9] as well by relying on the leximin criterion rather than the maximin for the SOF defined in the above proof. There are also many examples of SOFs satisfying Weak Pareto, IIA-WIS and Anonymity. Thus, in addition to these three axioms, one may add other requirements embodying various equity principles.[10]

[9]**Strong Pareto for SOFs**: $\forall x, y \in \mathbb{R}_+^{n\ell}, \forall \mathbf{R} \in \mathcal{R}^n$ if $\forall i \in N$, $x_i R_i y_i$, then $x\bar{R}(\mathbf{R})y$ and if, in addition, $\exists i \in N$, $x_i P_i y_i$, then $x\bar{P}(\mathbf{R})y$.

[10]Notice that Strong Pareto and Anonymity already entail a version of the Suppes grading principle: for all $\mathbf{R} \in \mathcal{R}^n$, all x, y, if there are i, j such that $R_i = R_j$, $x_i P_i y_j$ and $x_j P_i y_i$, and for $h \neq i, j$, $x_h = y_h$, then $x\bar{P}(\mathbf{R})y$. Notice also that it is easy to construct SOFs satisfying Strong Pareto, IIA-WIS (or IIA-ISPω_0), Anonymity and the following version of the Hammond equity axiom (Hammond, 1976), which is similar to the Equal Treatment of Equals axiom for ARs: for all $\mathbf{R} \in \mathcal{R}^n$, and all $x, y \in \mathbb{R}_+^{n\ell}$, if there are i, j such that $R_i = R_j$, $y_i P_i x_i P_i x_j P_i y_j$, and for $h \neq i, j$, $x_h = y_h$, then $x\bar{P}(\mathbf{R})y$.

4 Allocation Rules Need Marginal Rates of Substitution

For allocation rules, we first obtain a parallel to Arrow's theorem, although, strictly speaking, there is a possibility result even with the strongest form of IIA and Anonymity.

Proposition 6 *If S satisfies Pareto, IIA and Anonymity, then*

$$\forall \mathsf{R} \in \mathcal{R}^n,\ S(\mathsf{R}) = \{x \in \mathsf{R}_+^{nl} | \exists i \in N, x_i = \omega\}.$$

If S satisfies Pareto, IIA and is essentially single-valued, then it is dictatorial.

Proof. By Lemma 3, IIA⇔IP. By Pareto and IP, $S(\mathsf{R})$ must contain only allocations such that

$$\exists i \in N, x_i = \omega$$

because for any other allocation y, one can find R' such that $y \notin E(\mathsf{R})$.

By IP, for all $\mathsf{R}, \mathsf{R}' \in \mathcal{R}^n$,

$$\{i \in N | \exists x \in S(\mathsf{R}), x_i = \omega\} = \{i \in N | \exists x \in S(\mathsf{R}'), x_i = \omega\}.$$

Therefore Anonymity requires $\{i \in N | \exists x \in S(\mathsf{R}), x_i = \omega\} = N$, whereas essential single-valuedness requires $\{i \in N | \exists x \in S(\mathsf{R}), x_i = \omega\} = i_0$ for some fixed i_0. ■

It can also be immediately deduced from the proof that there does not exist an AR satisfying Pareto, IIA, and Equal Treatment of Equals.

The next result is that with IIA-MRS, Equal Treatment of Equals becomes attainable, but there remains a difficulty with essential single-valuedness.

Proposition 7 *There exists an AR satisfying Pareto, IIA-MRS, Anonymity, and Equal Treatment of Equals. There exists a non-dictatorial and essentially single-valued AR satisfying Pareto, IIA-MRS. However, there does not exist an essentially single-valued AR satisfying Pareto, IIA-MRS and Anonymity.*

Proof. The first possibility is illustrated by the Egalitarian Walrasian AR S_W, defined as follows: $x \in S_W(\mathsf{R})$ if $x \in F$ and there is $p \in \mathsf{R}_{++}^{\ell}$ such that for all $i \in N$,

$$\forall y \in \mathsf{R}_+^{\ell},\ p \cdot y \leq p \cdot \omega / n \Rightarrow x_i R_i y.$$

The second possibility is illustrated by the AR S_{p_0} defined for a given $p_0 \in \mathsf{R}^{\ell}_{++}$ as follows: if $p_0 \in C(\omega, R_1)$, then $S(\mathsf{R}) = \{x\}$ where $x_1 = \omega$, and otherwise, $S(\mathsf{R}) = \{y\}$ where $y_2 = \omega$.

For the impossibility, recall that by essential single-valuedness of S and Lemma 3, IIA-MRS⇔IP-MRS. Let $\mathcal{R}^*$ be the subset of $\mathcal{R}$ such for all $R \in \mathcal{R}^*$, R is differentiable, and moreover:

$$\forall z \in \mathsf{R}^{\ell}_{++},\ z' \in \mathsf{R}^{\ell}_{+} \hat{\mathsf{A}} \mathsf{R}^{\ell}_{++},\ zPz'.$$

Let $\mathsf{R} \in \mathcal{R}^{*\mathsf{n}}$ be given. Assume that there is $x \in S(\mathsf{R}) \setminus S_W(\mathsf{R})$. By Pareto $x \in E(\mathsf{R})$. Hence, we have $x_i \in \mathsf{R}^{\ell}_{++}$ or $x_i = 0$ for all i, and there is a shadow price vector $p \in \mathsf{R}^{\ell}_{++}$ such that

$$\forall i \in N,\ C(x_i, R_i) = \{p\} \text{ or } x_i = 0.$$

For this p, define $R^p \in \mathcal{R}$ by

$$\forall z, z' \in \mathsf{R}^{\ell}_{+},\ zR^p z' \Leftrightarrow p \cdot z \geq p \cdot z'.$$

Let $\mathsf{R}^p = (R^p, ..., R^p) \in \mathcal{R}^n$. By IP-MRS, $x \in S(\mathsf{R}^p)$. Since $x \notin S_W(\mathsf{R})$, there exist i, j, $x_i P^p x_j$, in contradiction to Equal Treatment of Equals. (Recollect that, by Lemma 1, essential single-valuedness and Anonymity imply Equal Treatment of Equals.) As a consequence, $S(\mathsf{R}) \subset S_W(\mathsf{R})$.

Assume there is $x \in S_W(\mathsf{R}) \setminus S(\mathsf{R})$. For all $i \in N$, let $R'_i \in \mathcal{R}^*$ be homothetic and strictly convex preferences satisfying

$$C(x_i, R'_i) = C(x_i, R_i).$$

Let $\mathsf{R}' = (R'_1, ..., R'_n) \in \mathcal{R}^{*\mathsf{n}}$. We have $x \in S_W(\mathsf{R}')$. Moreover, by Th. 1 in Eisenberg (1961), all allocations in $S_W(\mathsf{R}')$ are Pareto-indifferent. By strict convexity of preferences, one therefore has $S_W(\mathsf{R}') = \{x\}$. Since $S(\mathsf{R}') \subset S_W(\mathsf{R}')$, we have $S(\mathsf{R}') = \{x\}$. By IP-MRS, $x \in S(\mathsf{R})$, which is a contradiction. Therefore $S_W(\mathsf{R}) \subset S(\mathsf{R})$.

In conclusion, $S(\mathsf{R}) = S_W(\mathsf{R})$ for any profile $\mathsf{R} \in \mathcal{R}^{*\mathsf{n}}$. But we can find a profile $\mathsf{R} \in \mathcal{R}^{*\mathsf{n}}$ such that $S_W(\mathsf{R})$ contains two allocations x, y, and there exists i with $x_i P_i y_i$. This contradicts essential single-valuedness. ■

One can adapt the proof in order to show that the impossibility would not be removed by considering ε-neighborhoods instead of MRS, for ε small enough. Only with IIA-ISFA do we really obtain a full possibility result.

Proposition 8 *There exists an essentially single-valued AR satisfying Pareto, IP-ISFA and Anonymity.*

Proof. Let $\mathsf{R} \in \mathcal{R}^n$, $\omega \in \mathsf{R}^{\ell}_{++}$ be given. One defines S by: $x \in S(\mathsf{R})$ if $x \in E(\mathsf{R})$ and there is $\alpha \in \mathsf{R}_+$ such that for all $i \in N$,

$$x_i I_i \alpha\omega.$$

It obviously satisfies Pareto and Anonymity. To check that it satisfies IP-ISFA, notice that necessarily $\alpha < 1$, so that $\alpha\omega \in \Omega(\omega)$. ■

Notice that the AR described in the proof also satisfies Equal Treatment of Equals (by Lemma 1). And there are many examples of ARs satisfying Pareto, Anonymity and IP-WIS. In fact all the main ARs from the theory of fair allocation satisfy IP-WIS.

5 Conclusion

In a framework with purely ordinal, non-comparable preferences, a satisfactory social ordering function requires, when ranking any pair of allocations, information about the shape of indifference curves that goes well beyond purely local data such as marginal rates of substitution and preferences in ε-neighborhood. This is the first lesson of this paper. The second is that even for less demanding allocation rules, it is also necessary to introduce more information than what is allowed in the Arrow independence of irrelevant alternatives. The third is that, nonetheless, a purely local information such as marginal rates of substitution is sufficient for allocation rules, whereas it is not for social ordering functions.

We hope that our paper, more broadly, contributes to clarifying the informational foundations in the theory of social choice and in the theory of fair allocation, and also to clarifying the links and differences between these two theories.

There are limits to our work which should be emphasized here, and call for further research. First, we study a particular model, and it would be worth analyzing the same issues in other models such as the standard abstract model of social choice or other economic models, in particular models with public goods (the case of consumption externalities in our model could also be subsumed under the case of public goods). Second, the information about indifference curves is a complex set of objects and our analysis is far from being exhaustive on the pieces of data which can extracted from this set. For instance, it would be nice to have a measure of the degree to which a given piece of information is local. Third, there may be other kinds of interesting additional information. For instance, Roberts (1980) has considered introducing information about utilities and about non-local preferences at the same time, and was able to characterize the Nash social welfare function on this basis. There certainly are many avenues of research along these lines.

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7 Appendices

7.1 Proof of Proposition 2

The proof of Proposition 2 relies on the following lemmas.

Let $Y \subset X$ be a given finite subset of X. Let $i \in N$ be given. Let $Y_i = \{y_i \in \mathsf{R}_+^\ell |\ \exists y_{-i} \in \mathsf{R}_+^{(n-1)\ell}, (y_i, y_{-i}) \in Y\}$. For each $y_i \in Y_i$, let $Q(y_i) \subset \mathsf{R}_{++}^\ell$ be given. We say that the set Y_i *satisfies the supporting condition with the supporting price vectors* $\{Q(y_i)|\ y_i \in Y_i\}$ if for all $y_i \in Y_i$, all $q \in Q(y_i)$, and all $y_i' \in Y_i$ with $y_i' \neq y_i$, $q \cdot y_i < q \cdot y_i'$. Let

$$\mathcal{R}(Y_i, \{Q(y_i)|\ y_i \in Y_i\}) = \{R_i \in \mathcal{R}|\ \forall y_i \in Y_i, C(y_i, R_i) = Q(y_i)\}.$$

The set of all preorderings on Y_i is denoted by $\mathcal{O}(Y_i)$. For any $R_i \in \mathcal{R}$, $R_i|_{Y_i}$ denotes the restriction of R_i on Y_i[11]. For any $\mathcal{R}' \subset \mathcal{R}$, let $\mathcal{R}'|_{Y_i} = \{R_i|_{Y_i} \mid R_i \in \mathcal{R}'\}$. For any $x_i \in X$ and any $R_i \in \mathcal{R}$, let $U(x_i, R_i) = \{\ x_i' \in X \mid x_i'\ R_i\ x_i\ \}$ denote the (weak) upper contour set of x_i for R_i.

Lemma 4 *If a finite set $Y_i \subset \mathsf{R}_+^\ell$ satisfies the supporting condition with the supporting price vectors $\{Q(y_i)|\ y_i \in Y_i\}$, then $\mathcal{R}(Y_i, \{Q(y_i)|y_i \in Y_i\})|_{Y_i} = \mathcal{O}(Y_i)$.*

Proof. We need to show that $\mathcal{O}(Y_i) \subseteq \mathcal{R}(Y_i, \{Q(y_i)|y_i \in Y_i\})|_{Y_i}$. Let $R' \in \mathcal{O}(Y_i)$ be any preordering on Y_i. Construct a preordering $R_i \in \mathcal{R}$ so that the upper contour set of each $y_i \in Y_i$ is defined as follows. Let $x_i \in Y_i$ be such that for all $y_i \in Y_i$, $y_i\ R_i'\ x_i$. Let $Y_i^1 = \{y_i \in Y_i \mid y_i\ I_i'\ x_i\}$. Let

$$U(x_i, R_i) = \bigcap_{y_i \in Y_i^1} \bigcap_{q \in Q(y_i)} \{x_i' \in \mathsf{R}_+^\ell \mid q \cdot x_i' \geq q \cdot y_i\}$$

Let $I(x_i, R_i)$ be the boundary of $U(x_i, R_i)$. Clearly, for all $y_i \in Y_i^1$, $C(y_i, R_i) = Q(y_i)$. We also have that for all $y_i \in Y_i \backslash Y_i^1$, and for all $x_i' \in I(x_i, R_i)$, $y_i\ P_i\ x_i'$. Given $\varepsilon > 0$, let $\varepsilon U(x_i, R_i) = \{x_i' \in \mathsf{R}_+^\ell \mid \exists a_i \in U(x_i, R_i),\ x_i' = \varepsilon a_i\}$, and let $\varepsilon I(x_i, R_i)$ be the boundary of $\varepsilon U(x_i, R_i)$. For sufficiently small ε, we have that for all $y_i \in Y_i \backslash Y_i^1$, and for all $x_i' \in \varepsilon I(x_i, R_i)$, $y_i P_i x_i'$. Let $z_i \in Y_i \backslash Y_i^1$ be such that for all $y_i \in Y_i \backslash Y_i^1$, $y_i\ R_i'\ z_i$. Let $Y_i^2 = \{y_i \in Y_i \backslash Y_i^1 \mid y_i\ I_i'\ z_i\}$. Let

$$U(z_i, R_i) = \varepsilon U(x_i, R_i) \bigcap \left(\bigcap_{y_i \in Y_i^2} \bigcap_{q \in Q(y_i)} \{x_i' \in \mathsf{R}_+^\ell \mid q \cdot x_i' \geq q \cdot y_i\} \right)$$

[11] Namely, $R_i|_{Y_i}$ is the preordering on Y_i such that for all $x_i, y_i \in Y_i$, $x_i\ R_i|_{Y_i}\ y_i \iff x_i R_i y_i$.

Let $I(z_i, R_i)$ be the boundary of $U(z_i, R_i)$. By definition, for all $y_i \in Y_i^2$, $C(y_i, R_i) = Q(y_i)$. We have that for all $y_i \in Y_i \backslash (Y_i^1 \cup Y_i^2)$, and for all $x_i' \in I(z_i, R_i)$, $y_i \; P_i \; x_i'$. In the same way as above, we can construct the upper contour set of each $y_i \in Y_i \backslash (Y_i^1 \cup Y_i^2)$. By its construction, $R_i \in \mathcal{R}(Y_i, \{Q(y_i) | y_i \in Y_i\})$ and $R_i|_{Y_i} = R'$. Thus, $R' \in \mathcal{R}(Y_i, \{Q(y_i) | y_i \in Y_i\})|_{Y_i}$. ■

Let $\bar{R}$ be a social ordering function. Let $Y \subseteq X$ and $\mathcal{R}' \subseteq \mathcal{R}^n$ be given. We say that agent $i_0 \in N$ is a *local dictator* for $\bar{R}$ over $(Y, \mathcal{R}')$ if for all $x, y \in Y$, and all $\mathsf{R} \in R'$, $x_{i_0} P_{i_0} y_{i_0}$ implies $x \bar{P}(\mathsf{R}) y$.

Lemma 5 *Let $\bar{R}$ be a social ordering function satisfying Weak Pareto and IIA-MRS. Let $Y \subset X$ be a finite subset of X such that $|Y| \geq 3$.*[12] *Suppose that for all $i \in N$, Y_i satisfies the supporting condition with the supporting price vectors $\{Q(y_i)| \; y_i \in Y_i\}$. Then, there exists a local dictator $i_0 \in N$ for $\bar{R}$ over $(Y, \prod_{i \in N} \mathcal{R}(Y_i, \{Q(y_i)| \; y_i \in Y_i\}))$.*

Proof. For all $\mathsf{R}, \mathsf{R}' \in \prod_{i \in N} \mathcal{R}(Y_i, \{Q(y_i)| \; y_i \in Y_i\})$, all $y \in Y$, and all $i \in N$, $C(y_i, R_i) = C(y_i, R_i')$. Since $\bar{R}$ satisfies IIA-MRS, we have that for all $x, y \in Y$, and all $\mathsf{R}, \mathsf{R}' \in \prod_{i \in N} \mathcal{R}(Y_i, \{Q(y_i)| \; y_i \in Y_i\})$, if for all $i \in N$, $x_i R_i y_i \Leftrightarrow x_i R_i' y_i$, then $x \bar{R}(\mathsf{R}) y \Leftrightarrow x \bar{R}(\mathsf{R}') y$. By Lemma 4, for all $i \in N, \mathcal{R}(Y_i, \{Q(y_i) | y_i \in Y_i\})|_{Y_i} = \mathcal{O}(Y_i)$. Hence, by Arrow's Theorem, there exists a local dictator for $\bar{R}$ over $(Y, \prod_{i \in N} \mathcal{R}(Y_i, \{Q(y_i)| \; y_i \in Y_i\}))$. ■

We say that a subset Y of X is *free for agent i* if $\mathcal{R}|_{Y_i} = \mathcal{O}(Y_i)$. It is *free* if it is free for all $i \in N$. If Y contains two elements, it is a *free pair.* If Y contains three elements, it is a *free triple.* Note that a set $\{x, y\}$ is a free pair for $i \in N$ if and only if for some $k, k' \in \{1, \cdots, \ell\}$, $x_{ik} > y_{ik}$ and $y_{ik'} > x_{ik'}$. Given two consumption bundles $x_i, y_i \in \mathsf{R}_+^\ell$, define $x_i \wedge y_i \in \mathsf{R}_+^\ell$ as $(x_i \wedge y_i)_k = \min\{x_{ik}, y_{ik}\}$ for all $k \in \{1, \cdots, \ell\}$.

Lemma 6 *Let $\bar{R}$ be a social ordering function satisfying Weak Pareto and IIA-MRS. If $\{x, y\} \subset X$ is a free pair, then there exists a local dictator for $\bar{R}$ over $(\{x, y\}, \mathcal{R}^n)$.*

Proof. Let $\bar{R}$ be a social ordering function satisfying Weak Pareto and IIA-MRS. Let $\{x, y\} \subset X$ be a free pair. Let

$$\begin{aligned} K_1 &= \{k \in \{1, \cdots, \ell\} \mid x_{ik} > y_{ik}\} \\ K_2 &= \{k \in \{1, \cdots, \ell\} \mid x_{ik} < y_{ik}\} \end{aligned}$$

[12] Given a set A, $|A|$ denotes the cardinality of A.

Since $\{x, y\}$ is a free pair, $K_1, K_2 \neq \emptyset$.
Step 1 : For each $i \in N$, we define two consumption bundles $z_i, w_i \in X$ as follows:

$$z_i = x_i \wedge y_i + \frac{1}{2}\left\{\frac{2}{3}(x_i - x_i \wedge y_i) + \frac{1}{3}(y_i - x_i \wedge y_i)\right\} \quad (1)$$

$$w_i = x_i \wedge y_i + \frac{1}{2}\left\{\frac{1}{3}(x_i - x_i \wedge y_i) + \frac{2}{3}(y_i - x_i \wedge y_i)\right\} \quad (2)$$

Figure 1 illustrates the bundles $x_i, y_i, x_i \wedge y_i, z_i, w_i$, and also b_i, v_i, t_i, which are defined in the next step. Let $q \in \mathsf{R}^\ell_{++}$. Then, $q \cdot y_i < q \cdot w_i$ if and only if

$$\frac{2}{3}\sum_{k \in K_2} q_k(y_{ik} - x_{ik}) < \frac{1}{6}\sum_{k \in K_1} q_k(x_{ik} - y_{ik}) \quad (3)$$

Since $K_1 \neq \emptyset$, the right-hand-side of (3) can be arbitrarily large as $(q_k)_{k \in K_1}$ become large, $(q_k)_{k \in K_2}$ being constant. Hence, there exists a price vector $q(y_i) \in \mathsf{R}^\ell_{++}$ that satisfies inequality (3). With some calculations, it can be shown that $q(y_i) \cdot y_i < q(y_i) \cdot z_i$ and $q(y_i) \cdot y_i < q(y_i) \cdot x_i$.

Similarly, for each $a \in \{x_i, z_i, w_i\}$, we can find a price vector $q(a) \in \mathsf{R}^\ell_{++}$ such that for all $a' \in \{x_i, z_i, w_i, y_i\}$ with $a' \neq a$, $q(a) \cdot a < q(a) \cdot a'$. Hence, the set $Y_i^0 = \{x_i, z_i, w_i, y_i\}$ satisfies the supporting condition with the supporting price vectors $\{q(x_i), q(z_i), q(w_i), q(y_i)\}$.[13]

Let $z = (z_i)_{i \in N}$ and $w = (w_i)_{i \in N}$. Let $Y^0 = \{x, z, w, y\}$. By Lemma 5, there exists a local dictator $i_0 \in N$ for $\bar{R}$ over $(Y^0, \prod_{i \in N} \mathcal{R}(Y_i^0, \{q(x_i), q(z_i), q(w_i), q(y_i)\})$.
Step 2: We will show that agent i_0 is a local dictator for $\bar{R}$ over $(\{x, y\}, \mathcal{R}^n)$.

Suppose, on the contrary, that there exists a preference profile $\mathsf{R}^0 \in \mathcal{R}^n$ such that (i) $x_{i_0} P^0_{i_0} y_{i_0}$ and $y\bar{R}(\mathsf{R}^0)x$ or (ii) $y_{i_0} P^0_{i_0} x_{i_0}$ and $x\bar{R}(\mathsf{R}^0)y$. Without loss of generality, suppose that (i) holds. Let $Y^1 = \{z, w, y\}$. Since agent i_0 is the local dictator for $\bar{R}$ over $(Y^0, \prod_{i \in N} \mathcal{R}(Y_i^0, \{q(x_i), q(z_i), q(w_i), q(y_i)\})$, he is also the local dictator for $\bar{R}$ over $(Y^1, \prod_{i \in N} \mathcal{R}(Y_i^1, \{q(z_i), q(w_i), q(y_i)\})$. (Otherwise, by Lemma 5, there exists a local dictator $j \neq i_0$ for $\bar{R}$ over $(Y^1, \prod_{i \in N} \mathcal{R}(Y_i^1, \{q(z_i), q(w_i), q(y_i)\})$, and we can construct a preference profile $\mathsf{R} \in \prod_{i \in N} \mathcal{R}(Y_i^0, \{q(x_i), q(z_i), q(w_i), q(y_i)\}) \subset \prod_{i \in N} \mathcal{R}(Y_i^1, \{q(z_i), q(w_i), q(y_i)\})$ such that $z_{i_0} P_{i_0} w_{i_0}$ and $w_j P_j z_j$. Hence we must have $z\bar{P}(\mathsf{R})w$ and $w\bar{P}(\mathsf{R})z$, which is a contradiction.)

[13] With a slight abuse of notation, we write $\{q(w_i), q(z_i), q(y_i)\}$ for $\{\{q(w_i)\}, \{q(z_i)\}, \{q(y_i)\}\}$.

We define two allocations $v, t \in X$ in the following steps. Let $i \in N$. First, define $b_i \in \mathsf{R}^\ell_+$ as follows: If for all $q \in C(x_i, R_i^0)$, $q \cdot (y_i - x_i) \geq 0$, then let $b_i = y_i$. If for some $q \in C(x_i, R_i^0)$, $q \cdot (y_i - x_i) < 0$, then let $\theta > 0$ be a positive number such that for all $q \in C(x_i, R_i^0)$, $q \cdot \{y_i + \theta(y_i - x_i \wedge y_i) - x_i\} > 0$. Since $q \in \mathsf{R}^\ell_{++}$ by strict monotonicity of preferences, and $y_i - x_i \wedge y_i > 0$, such a number θ exists. Then, define $b_i = y_i + \theta(y_i - x_i \wedge y_i)$. By definition, $b_i > y_i$, and for all $q \in C(x_i, R_i^0)$, $q \cdot (b_i - x_i) > 0$. Let

$$v_i = b_i + 2(b_i - x_i \wedge y_i)$$

Then, $v_i > b_i > y_i$, and for all $q \in C(x_i, R_i^0)$, $q \cdot (v_i - x_i) > 0$.

Next, let

$$t_i = x_i \wedge y_i + \frac{1}{2}\left\{\frac{2}{3}(v_i - x_i \wedge y_i) + \frac{1}{3}(w_i - x_i \wedge y_i)\right\}$$

Then,

$$t_i = b_i + \frac{1}{6}(w_i - x_i \wedge y_i) > b_i$$

and for all $q \in C(x_i, R_i^0)$, $q \cdot x_i < q \cdot t_i$.

As in Step 1, we can find price vectors $q(v_i), q(t_i) \in \mathsf{R}^\ell_{++}$ such that $q(v_i) \cdot v_i < q(v_i) \cdot a$ for all $a \in \{x_i, z_i, w_i, t_i\}$, and $q(t_i) \cdot t_i < q(t_i) \cdot a$ for all $a \in \{x_i, z_i, w_i, v_i\}$.

On the other hand, because $v_i > y_i$ and $t_i > y_i$, we have that $q(z_i) \cdot z_i < q(z_i) \cdot a$ for all $a \in \{t_i, v_i\}$, and $q(w_i) \cdot w_i < q(w_i) \cdot a$ for all $a \in \{t_i, v_i\}$.

So far we have shown that
(i) the set $Y_i^1 = \{x_i, t_i, v_i\}$ satisfies the supporting condition with the supporting price vectors $\{C(x_i, R_i^0), q(t_i), q(v_i)\}$.
(ii) the set $Y_i^2 = \{z_i, w_i, t_i, v_i\}$ satisfies the supporting condition with the supporting price vectors $\{q(z_i), q(w_i), q(t_i), q(v_i)\}$.

Let $v = (v_i)_{i \in N}$ and $t = (t_i)_{i \in N}$. Let $Y^1 = \{x, t, v\}$ and $Y^2 = \{z, w, t, v\}$. By Lemma 5, there exist a local dictator $i_1 \in N$ for $\bar{R}$ over $(Y^1, \prod_{i \in N} \mathcal{R}(Y_i^1, \{C(x_i, R_i^0), q(t_i), q(v_i)\})$, and a local dictator $i_2 \in N$ for $\bar{R}$ over $(Y^2, \prod_{i \in N} \mathcal{R}(Y_i^2, \{q(z_i), q(w_i), q(t_i), q(v_i)\})$. Recall that agent $i_0 \in N$ is the local dictator for $\bar{R}$ over $(Y^0, \prod_{i \in N} \mathcal{R}(Y_i^0, \{q(z_i), q(w_i), q(y_i)\})$. Let $\mathsf{R}^1 \in \mathcal{R}^n$ be a preference profile such that for all $i \in N$, $C(x_i, R_i^1) = C(x_i, R_i^0)$, and for all $a_i \in \{t_i, v_i, w_i, y_i, z_i\}$, $C(a_i, R_i^1) = \{q(a_i)\}$, and such that

$$x_{i_0} P^1_{i_0} z_{i_0} P^1_{i_0} w_{i_0} P^1_{i_0} t_{i_0} P^1_{i_0} v_{i_0} P^1_{i_0} y_{i_0}$$

and for all $i \in N$ with $i \neq i_0$, $x_i P_i^1 v_i P_i^1 t_i P_i^1 w_i P_i^1 z_i P_i^1 y_i$

Since $\mathsf{R}^1 \in \prod_{i\in N} \mathcal{R}(Y_i^0, \{q(z_i), q(w_i), q(y_i)\}$, and agent i_0 is the local dictator for $\bar{R}$ over $(Y^0, \prod_{i\in N} \mathcal{R}(Y_i^0, \{q(z_i), q(w_i), q(y_i)\})$, we have $z\bar{P}(\mathsf{R}^1)w$. Because $\mathsf{R}^1 \in \prod_{i\in N} \mathcal{R}(Y_i^2, \{q(z_i), q(w_i), q(t_i), q(v_i)\}$, this implies that $i_0 = i_2$. Hence, we have $t\bar{P}(\mathsf{R}^1)v$. Since $\mathsf{R}^1 \in \prod_{i\in N} \mathcal{R}(Y_i^1, \{C(x_i, R_i^0), q(t_i), q(v_i)\}$, it follows that $i_0 = i_1$.

Let $\mathsf{R}^2 \in \mathcal{R}^n$ be a preference profile such that $x_{i_0} P_{i_0}^2 v_{i_0}$ and for all $i \in N$, $R_i^2|_{\{x_i, y_i\}} = R_i^0|_{\{x_i, y_i\}}$, and $C(x_i, R_i^2) = C(x_i, R_i^0), C(t_i, R_i^2) = \{q(t_i)\}, C(v_i, R_i^2) = \{q(v_i)\}$ and $C(y_i, R_i^2) = C(y_i, R_i^0)$. Since agent $i_0 \in N$ is the local dictator for $\bar{R}$ over $(Y^1, \prod_{i\in N} \mathcal{R}(Y_i^1, \{C(x_i, R_i^0), q(t_i), q(v_i)\})$, we have that $x\bar{P}(\mathsf{R}^2)v$. Recall that for all $i \in N$, $v_i > y_i$. Hence, by strict monotonicity of preferences, $v_i P_i^2 y_i$ for all $i \in N$. Because the social ordering function $\bar{R}$ satisfies Weak Pareto, we have $v\bar{P}(\mathsf{R}^2)y$. By transitivity of $\bar{R}$, $x\bar{P}(\mathsf{R}^2)y$. However, since $\bar{R}$ satisfies IIA-MRS, and $y\bar{R}(\mathsf{R}^0)x$, we must have $y\bar{R}(\mathsf{R}^2)x$. This is a contradiction. ■

Lemma 7 *Let $\bar{R}$ be a social ordering function satisfying Weak Pareto and IIA-MRS. If $\{x, y, z\} \subset X$ is a free triple, then there exists a local dictator for $\bar{R}$ over $(\{x, y, z\}, \mathcal{R}^n)$.*

Proof. By Lemma 6, there exist a local dictator i_0 over $(\{x, y\}, \mathcal{R}^n)$, , a local dictator i_1 over $(\{y, z\}, \mathcal{R}^n)$, and a local dictator i_2 over $(\{x, z\}, \mathcal{R}^n)$. Suppose that $i_0 \neq i_1$. Let $\mathsf{R} \in \mathcal{R}^n$ be a preference profile such that $x_{i_0} P_{i_0} y_{i_0}$, $y_{i_2} P_{i_2} z_{i_3}$, and $z_{i_1} P_{i_1} x_{i_1}$. Then, we have $x\bar{P}(\mathsf{R})y\bar{P}(\mathsf{R})z\bar{P}(\mathsf{R})x$, which contradicts the transitivity of $\bar{R}(\mathsf{R})$. Hence, we must have $i_0 = i_1$. By the same argument, we have $i_0 = i_1 = i_2$. ■

Proof of Proposition 2: Let $\bar{R}$ be a social ordering function satisfying Weak Pareto and IIA-MRS. By Lemma 6, for every free pair $\{x, y\} \subset X$, there exists a local dictator over $(\{x, y\}, \mathcal{R}^n)$. By Lemma 7 and Bordes and Le Breton (1989, Theorem 2), these dictators must be the same individual. Denote the individual by i_0. It remains to show that for any pair $\{x, y\}$ that is not free, i_0 is the local dictator over $(\{x, y\}, \mathcal{R}^n)$. Suppose, on the contrary, that there exist $\{x, y\} \subset X$ and $\mathsf{R} \in \mathcal{R}^n$ such that $\{x, y\}$ is not a free pair, and $x_{i_0} P_{i_0} y_{i_0}$ but $y\bar{R}(\mathsf{R})x$. Define $z_{i_0} \in \mathsf{R}_+^\ell$ as follows.
Case 1: $\{x, y\}$ is a free pair for i_0.
For all $\lambda \in]0, 1[$, $\{\lambda x + (1-\lambda)y, x\}$ and $\{\lambda x + (1-\lambda)y, y\}$ are free pairs for i_0. By continuity, there exists λ^* such that $x_{i_0} P_{i_0} [\lambda^* x_{i_0} + (1-\lambda^*)y_{i_0}] P_{i_0} y_{i_0}$. Then, let $z_{i_0} = \lambda^* x_{i_0} + (1-\lambda^*)y_{i_0}$..
Case 2: $\{x, y\}$ is not a free pair for i_0.

Then, for all $k \in \{1, \cdots, \ell\}$, $x_{i_0 k} \geq y_{i_0 k}$ with at least one strict inequality. Note that $y \neq 0$.
Case 2-1: There exists k' such that for all $k \in \{1, \cdots, \ell\}$ with $k \neq k'$, $x_{i_0 k} = y_{i_0 k}$ and $y_{i_0 k'} > 0$.
Then, $x_{i_0 k'} > y_{i_0 k'} > 0$. Given $\varepsilon > 0$, define $w_{i_0} \in \mathsf{R}^\ell_+$ as $w_{i_0 k'} = y_{i_0 k'}$ and for all $k \neq k', w_{i_0 k} = y_{i_0 k} + \varepsilon$. For sufficiently small ε, we have $x_{i_0} P_{i_0} w_{i_0} P_{i_0} y_{i_0}$. Given $\delta > 0$, define $t_{i_0} \in \mathsf{R}^\ell_+$ as $t_{i_0 k'} = w_{i_0 k'} - \delta$ and for all $k \neq k', t_{i_0 k} = w_{i_0 k}$. For sufficiently small δ, we have $x_{i_0} P_{i_0} t_{i_0} P_{i_0} y_{i_0}$. Moreover, $\{t, x\}$ and $\{t, y\}$ are free pairs for i_0. Then, let $z_{i_0} = t_{i_0}$.
Case 2-2: There exists k' such that for all $k \in \{1, \cdots, \ell\}$ with $k \neq k'$, $x_{i_0 k} = y_{i_0 k}$ and $y_{i_0 k'} = 0$.
Then, for all $k \in \{1, \cdots, \ell\}$ with $k \neq k'$, $x_{i_0 k} = y_{i_0 k} > 0$. Let $k'' \neq k'$. Given $\varepsilon > 0$, define $w_{i_0} \in \mathsf{R}^\ell_+$ as $w_{i_0 k''} = x_{i_0 k''} - \varepsilon$ and for all $k \neq k'', w_{i_0 k} = x_{i_0 k}$. For sufficiently small ε, we have $x_{i_0} P_{i_0} w_{i_0} P_{i_0} y_{i_0}$. Given $\delta > 0$, define $t_{i_0} \in \mathsf{R}^\ell_+$ as $t_{i_0 k'} = w_{i_0 k'} + \delta$ and for all $k \neq k', t_{i_0 k} = w_{i_0 k}$. For sufficiently small δ, we have $x_{i_0} P_{i_0} t_{i_0} P_{i_0} y_{i_0}$. Moreover, $\{t, x\}$ and $\{t, y\}$ are free pairs for i_0. Then, let $z_{i_0} = t_{i_0}$.
Case 2-3: There exists $k', k'' \in \{1, \cdots, \ell\}$ with $k' \neq k''$, $x_{i_0 k'} > y_{i_0 k'}$ and $x_{i_0 k''} > y_{i_0 k''}$.
Let k^* be such that $y_{i_0 k^*} > 0$. Given $\varepsilon > 0$, define $w_{i_0} \in \mathsf{R}^\ell_+$ as $w_{i_0 k^*} = y_{i_0 k^*} - \varepsilon$ and for all $k \neq k^*, w_{i_0 k} = x_{i_0 k}$. For sufficiently small ε, we have $x_{i_0} P_{i_0} w_{i_0} P_{i_0} y_{i_0}$. Let $k^{**} \neq k^*$. Given $\delta > 0$, define $t_{i_0} \in \mathsf{R}^\ell_+$ as $t_{i_0 k^{**}} = w_{i_0 k^{**}} + \delta$ and for all $k \neq k^{**}, t_{i_0 k} = w_{i_0 k}$. For sufficiently small δ, we have $x_{i_0} P_{i_0} t_{i_0} P_{i_0} y_{i_0}$. Moreover, $\{t, x\}$ and $\{t, y\}$ are free pairs for i_0. Then, let $z_{i_0} = t_{i_0}$.

Next, for each $i \neq i_0$, let $z_i \in \mathsf{R}^\ell_+$ be such that $\{z, x\}$ and $\{z, y\}$ are free pairs for i. By the same construction as above, we can find such $z_i \in \mathsf{R}^\ell_+$ for each i. Let $z = (z_i)_{i \in N} \in \mathsf{R}^{n\ell}_+$. Since i_0 is the dictator over all free pairs, we have that $x\bar{P}(\mathsf{R})z$ and $z\bar{P}(\mathsf{R})y$. By transitivity of $\bar{R}$, we have $x\bar{P}(\mathsf{R})y$, which contradicts the supposition that $y\bar{R}(\mathsf{R})x$.¥

7.2 Proof of Proposition 4

To prove the possibility result, let

$$\mathcal{R}^* = \{R \in \mathcal{R} \,|\, \forall x \in \mathsf{R}^\ell_{++}, y \in \mathsf{R}^\ell_+ \setminus \mathsf{R}^\ell_{++}, xPy\},$$

and say that a preference preordering $R \in \mathcal{R}$ is differentiable at zero if there exists an open set V containing 0 such that R is differentiable on $V \cap \mathsf{R}^\ell_+$.

If $\mathsf{R} \in \mathcal{R}^{*n}$, $x \in \mathsf{R}^{n\ell}_{++}$ and $y \in \mathsf{R}^{n\ell}_{+} \setminus \mathsf{R}^{n\ell}_{++}$, then there is $i \in N$ such that $x_i P_i y_i$. Define $\bar{R}$ as follows. One has $x\bar{R}(\mathsf{R})y \Leftrightarrow x_1 R_1 y_1$ whenever $\mathsf{R} \notin \mathcal{R}^{*n}$ or $x, y \in \mathsf{R}^{n\ell}_{+} \setminus \mathsf{R}^{n\ell}_{++}$, or $x, y \in \mathsf{R}^{n\ell}_{++}$ and R_1 is differentiable at zero. One has $x\bar{R}(\mathsf{R})y \Leftrightarrow x_2 R_2 y_2$ whenever $\mathsf{R} \in \mathcal{R}^{*n}$ and $x, y \in \mathsf{R}^{n\ell}_{++}$ and R_1 is not differentiable at zero. One has $x\bar{P}(\mathsf{R})y$ whenever $\mathsf{R} \in \mathcal{R}^{*n}$ and $x \in \mathsf{R}^{n\ell}_{++}$ and $y \in \mathsf{R}^{n\ell}_{+} \setminus \mathsf{R}^{n\ell}_{++}$.

To check Weak Pareto, assume that $x_i P_i y_i$ for all i. Then, when $\mathsf{R} \in \mathcal{R}^{*n}$ it is impossible to have $y \in \mathsf{R}^{n\ell}_{++}$ and $x \in \mathsf{R}^{n\ell}_{+} \setminus \mathsf{R}^{n\ell}_{++}$, so that in all possible cases, necessarily $x\bar{P}(\mathsf{R})y$. To check IIA-ISFA, notice that when $x, y \in \mathsf{R}^{n\ell}_{++}$, $\Omega(\omega(x, y))$ contains a neighborhood of 0, so that by changing individual preferences on $\mathsf{R}^{\ell}_{+} \setminus \Omega(\omega(x, y))$, one cannot change the fact that R_1 is differentiable at zero or not. When $y \in \mathsf{R}^{n\ell}_{+} \setminus \mathsf{R}^{n\ell}_{++}$, one has $x\bar{P}(\mathsf{R})y$ if $x \in \mathsf{R}^{n\ell}_{++}$, and $x\bar{R}(\mathsf{R})y \Leftrightarrow x_1 R_1 y_1$ if $x \in \mathsf{R}^{n\ell}_{+} \setminus \mathsf{R}^{n\ell}_{++}$, which means that individual preferences on $\Omega(\omega(x, y))$ (actually, on $\{x, y\}$) fully determine $\bar{R}(\mathsf{R})$ on $\{x, y\}$. Now, one sees that no agent is a dictator, for all profiles, over all allocations in A. Finally, it remains to check that transitivity is always obtained. If $\mathsf{R} \notin \mathcal{R}^{*n}$, this is due to transitivity of R_1. If $\mathsf{R} \in \mathcal{R}^{*n}$, and R_1 is differentiable at zero, transitivity is similarly guaranteed over $\mathsf{R}^{n\ell}_{+} \setminus \mathsf{R}^{n\ell}_{++}$ and over $\mathsf{R}^{n\ell}_{++}$, while strict social preference always occurs for $\mathsf{R}^{n\ell}_{++}$ against $\mathsf{R}^{n\ell}_{+} \setminus \mathsf{R}^{n\ell}_{++}$. If $\mathsf{R} \in \mathcal{R}^{*n}$, and R_1 is not differentiable at zero, transitivity is guaranteed over $\mathsf{R}^{n\ell}_{+} \setminus \mathsf{R}^{n\ell}_{++}$ by transitivity of R_1, and over $\mathsf{R}^{n\ell}_{++}$ by transitivity of R_2, while strict social preference always occurs for $\mathsf{R}^{n\ell}_{++}$ against $\mathsf{R}^{n\ell}_{+} \setminus \mathsf{R}^{n\ell}_{++}$.

In order to prove the impossibility, it is convenient to consider different possible sizes of the population. Let $\lambda \geq 1$ be such that IIA-ISEFA is satisfied by $\bar{R}$ with respect to it.

Case 1: $n = 2$. Consider the bundles $x = (8, 1/(2\lambda), 0, ...)$, $y = (12, 1/(2\lambda), 0, ...)$, $z = (1/(2\lambda), 12, 0, ...)$, $w = (1/(2\lambda), 8, 0, ...)$. Let preferences R_1 and R_2 be defined as follows. On the subset

$$S_1 = \{v \in R^{\ell}_{+} | \forall i \in \{3, ..., \ell\}, v_i = 0 \text{ and } v_2 \leq \min\{v_1, 1\}\}$$

one has

$$vR_1v' \Leftrightarrow v_1 + 2v_2 \geq v'_1 + 2v'_2,$$

and on the subset

$$S_2 = \{v \in R^{\ell}_{+} | \forall i \in \{3, ..., \ell\}, v_i = 0 \text{ and } v_1 \leq \min\{v_2, 1\}\},$$

one has

$$vR_1v' \Leftrightarrow 2v_1 + v_2 \geq 2v'_1 + v'_2.$$

Since

$$w_1 + (1 - w_1) + 2\left[w_2 - 2(1 - w_1)\right] > x_1 + 2x_2$$

and

$$2\left[y_1 - 2(1 - y_2)\right] + y_2 + (1 - y_2) > 2z_1 + z_2,$$

it is possible to complete the definition of R_1 such that wP_1x and yP_1z. Then define R_2 so that it coincides with R_1 on $S_1 \cup S_2$. Similarly, it is possible to complete the definition of R_2 such that xP_2w and zP_2y. Figure 2 illustrates this construction (for $\lambda = 1$).

If the profile of preferences is $\mathsf{R} = (R_1, R_2)$, by Weak Pareto one has:

$$(y, x)\bar{P}(\mathsf{R})(z, w) \text{ and } (w, z)\bar{P}(\mathsf{R})(x, y).$$

If the profile of preferences is $\mathsf{R}' = (R_1, R_1)$, by Anonymity one has:

$$(y, x)\bar{I}(\mathsf{R}')(x, y) \text{ and } (w, z)\bar{I}(\mathsf{R}')(z, w).$$

By IIA-ISEFA, since R_1 and R_2 coincide on $S_1 \cup S_2$,

$$\begin{aligned} (y, x)\bar{I}(\mathsf{R}')(x, y) &\Leftrightarrow (y, x)\bar{I}(\mathsf{R})(x, y) \\ \text{and } (w, z)\bar{I}(\mathsf{R}')(z, w) &\Leftrightarrow (w, z)\bar{I}(\mathsf{R})(z, w). \end{aligned}$$

By transitivity, one gets $(x, y)\bar{P}(\mathsf{R})(x, y)$, which is impossible.

Case 2: $n = 3$. Consider the bundles $x = (8, 1/(3\lambda), 0, ...)$, $y = (12, 1/(3\lambda), 0, ...)$, $t = (10, 1/(3\lambda), 0, ...)$, $z = (1/(3\lambda), 12, 0, ...)$, $w = (1/(3\lambda), 8, 0, ...)$, $r = (1/(3\lambda), 10, 0, ...)$. Let preferences R_1, R_2 and R_3 be defined as above on the subset $S_1 \cup S_2$. And complete their definition so that yP_1z, wP_1x, tP_2r, zP_2y, xP_3w, rP_3t.

If the profile of preferences is $\mathsf{R} = (R_1, R_2, R_3)$, by Weak Pareto one has:

$$(y, t, x)\bar{P}(\mathsf{R})(z, r, w) \text{ and } (w, z, r)\bar{P}(\mathsf{R})(x, y, t).$$

If the profile of preferences is $\mathsf{R}' = (R_1, R_1, R_1)$, by Anonymity one has:

$$(y, t, x)\bar{I}(\mathsf{R}')(x, y, t) \text{ and } (w, z, r)\bar{I}(\mathsf{R}')(z, r, w).$$

By IIA-ISEFA, since R_1, R_2 and R_3 coincide on S_1, and S_2 respectively,

$$\begin{aligned} (y, t, x)\bar{I}(\mathsf{R}')(x, y, t) &\Leftrightarrow (y, t, x)\bar{I}(\mathsf{R})(x, y, t) \\ \text{and } (w, z, r)\bar{I}(\mathsf{R}')(z, r, w) &\Leftrightarrow (w, z, r)\bar{I}(\mathsf{R})(z, r, w). \end{aligned}$$

By transitivity, one gets $(x,y,t)\bar{P}(\mathsf{R})(x,y,t)$, which is impossible.
Case 3: $n = 2k$. Partition the population into k pairs, and construct an argument similar to the case $n = 2$, with the bundles $x = (8, 1/(n\lambda), 0, ...)$, $y = (12, 1/(n\lambda), 0, ...)$, $z = (1/(n\lambda), 12, 0, ...)$, $w = (1/(n\lambda), 8, 0, ...)$, and the allocations $(y,x,y,x,...)$, $(x,y,x,y,...)$, $(z,w,z,w,...)$ and $(w,z,w,z,...)$.
Case 4: $n = 2k+1$. Partition the population into $k-1$ pairs and one triple, and construct an argument combining the cases $n = 2$ and $n = 3$, with the bundles $x = (8, 1/(n\lambda), 0, ...)$, $y = (12, 1/(n\lambda), 0, ...)$, $t = (10, 1/(n\lambda), 0, ...)$, $z = (1/(n\lambda), 12, 0, ...)$, $w = (1/(n\lambda), 8, 0, ...)$, $r = (1/(n\lambda), 10, 0, ...)$, and the allocations $(y,x,y,x,...,y,t,x)$, $(x,y,x,y,...,x,y,t)$, $(z,w,z,w,...z,r,w)$ and $(w,z,w,z,...,w,z,r)$. ■

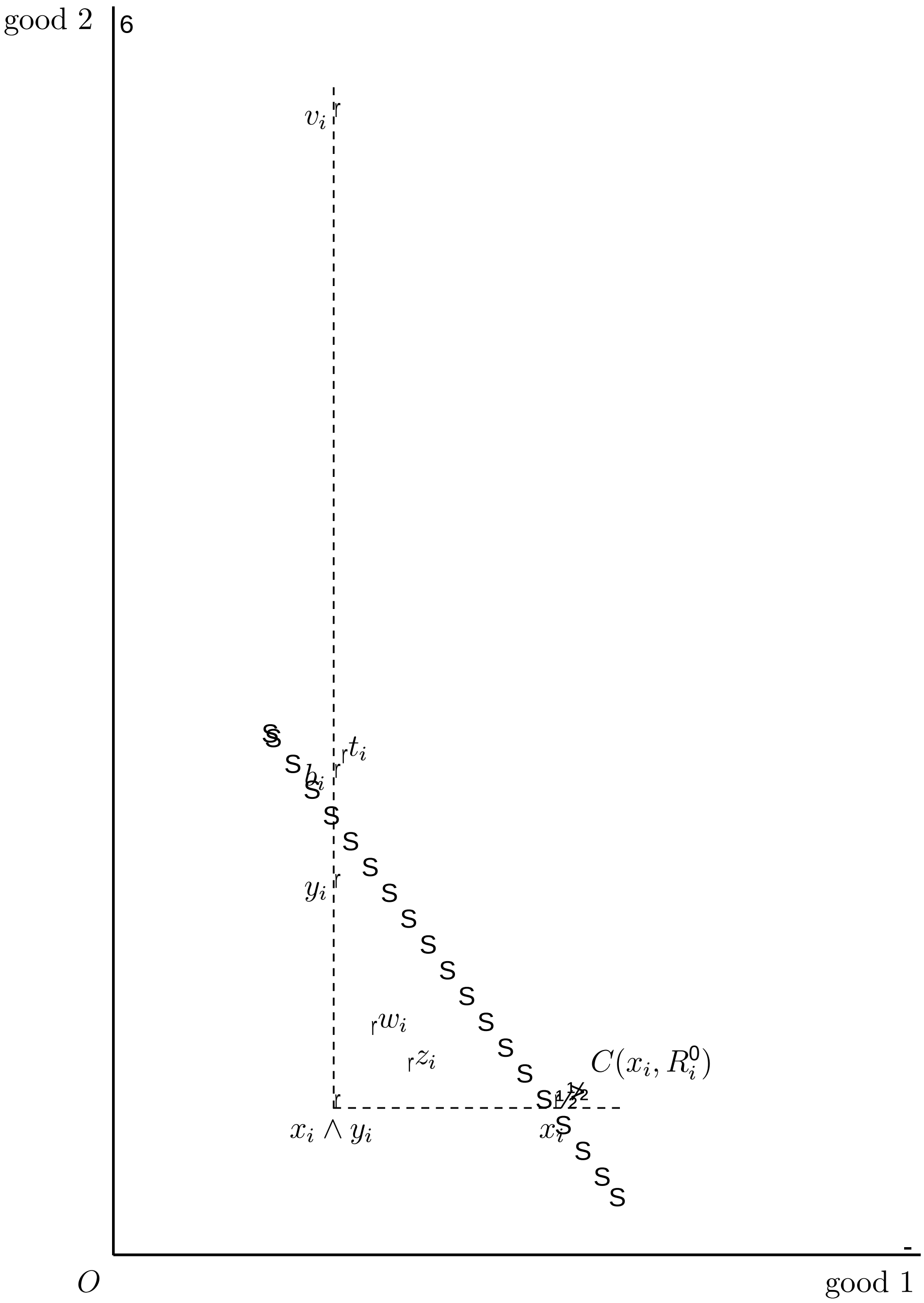
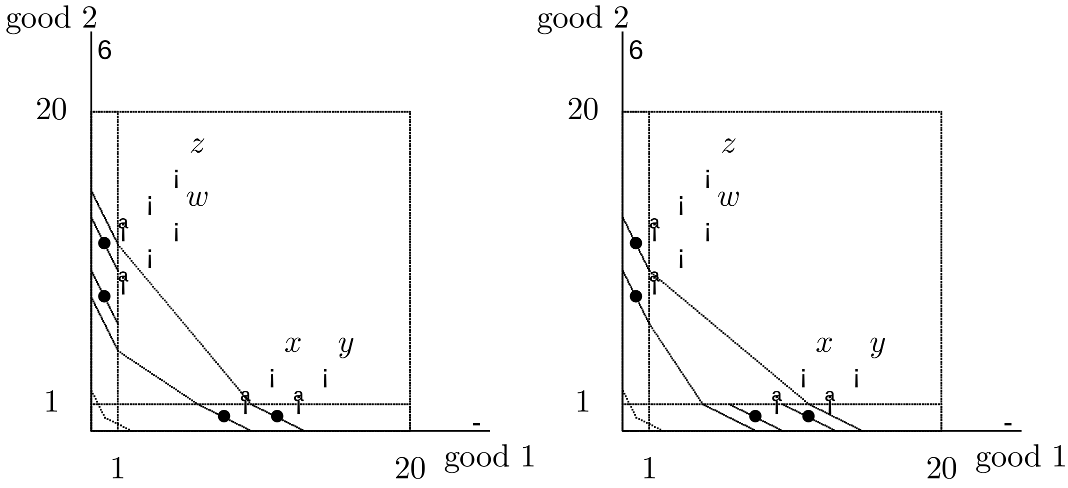


Figure 1: Proof of Lemma 6



Preferences R_1 Preferences R_2

Figure 2: Proof of Proposition 4