On the Possibility of Continuous, Paretian and Egalitarian Evaluation of Infinite Utility Streams*

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1. Introduction

There is a strong utilitarian tradition of treating otherwise equal generations equally. In the parlance of Henry Sidgwick (1908, p.414), "the time at which a man exists cannot affect the value of his happiness from a universal point of view; and that the interests of posterity must concern a Utilitarian as much as those of his contemporaries ... ." However, a serious doubt was raised by Tjalling Koopmans (1960) on the sustainability of this viewpoint by showing that the rational, continuous, and stationary evaluation of infinite allocation programs cannot but exhibit a phenomenon which Koopmans christened *impatience*, viz., the preference for advancement along the time axis of an outcome yielding higher utility vis-à-vis another outcome yielding lower utility. This intriguing thesis was elaborated further by Peter Diamond (1965) into a couple of general impossibility theorems to the effect that there exists no social evaluation ordering over the set of infinite utility streams which satisfies the Pareto principle, the equity principle à la Sidgwick, and the technical axiom of continuity. Note that Diamond's continuity axiom was defined with respect to the sup topology, on the one hand, and the product topology, on the other. Some of the subsequent work along the Koopmans-Diamond line such as Donald Campbell (1985), Luc Lauwers (1997), Tomoichi Shinotsuka (1998), and Lars-Gunnar Svensson (1980) examined the sensitivity of Diamond's theorems on the choice of underlying topology.

A recent paper by Kaushik Basu and Tapan Mitra (2003) critically reexamined one of Diamond's impossibility theorems, where their critical axe fell exclusively on his continuity axiom which "is a technical axiom (in contrast to the other two axioms) [Basu and Mitra (2003, p.1557)]." Note, however, that their paper retained the numerical representability of the social evaluation ordering, which, in itself, is a highly restrictive technical requirement. As a matter of fact, their impossibility theorems seem to have much to do with this retained technical axiom of numerical representability.

The present paper also reexamines Diamond's impossibility theorems, but the focus of our reexamination is completely different from that of Basu and Mitra (2003). Recollect that the equity principle à la Sidgwick is purely procedural in nature, and it does not embody any preference for egalitarian distribution of utilities among generations. Two versions of distributional egalitarianism in the spirit of Anthony Atkinson (1970) and Amartya Sen (1997) are introduced, and their compatibility with the strong Pareto principle in the presence of weaker version of the continuity axiom is examined. Unlike Basu and Mitra (2003), we do not require numerical representability of social evaluation relation. As a matter of fact, the social evaluation relation is assumed to satisfy neither completeness nor transitivity, so that it is not numerically representable in general. These differences notwithstanding, it is shown that the non-existence results strenuously come to the fore.

Apart from this Introduction, this paper consists of four sections. Section 2 introduces our basic model and axioms. Section 3 introduces the first distributional equity axiom in the spirit of the Pigou-Dalton transfer principle, and establishes the first impossibility theorem in the presence of upper semi-continuity with respect to the sup topology and acyclicity of the social evaluation relation. Section 4 introduces the second distributional equity axiom in the spirit of the Lorenz domination principle, and establishes the second impossibility theorem in the presence of upper semi-continuity with respect to the sup topology and asymmetry of the social evaluation relation. A generalization of the second impossibility theorem is also shown to hold by establishing that the sup topology can be generalized without upsetting the validity of the impossibility theorem. Section 5 concludes this paper with some interpretative remarks.

2. Basic Model and Axioms

Let $\mathbb{R}$ and $\mathbb{N}$ denote the set of all real numbers and the set of all positive integers, respectively. The set of all non-negative real numbers is denoted by $\mathbb{R}_+$. For the sake of simplicity, we assume that $X := \mathbb{R}_+^\infty$ is the set of all infinite utility streams, viz., $u = (u_1, u_2, ..., u_n, ...) \in X$ denotes an infinite sequence of utilities, where u_n denotes the utility of generation $n \in \mathbb{N}$. For all $u = (u_1, u_2, ..., u_n, ...)$, $v = (v_1, v_2, ..., v_n, ...) \in X$, $u \geq v$ means that $u_n \geq v_n$ for all $n \in \mathbb{N}$; $u > v$ means that $u \geq v$ and $u \neq v$; $u >> v$ means that $u_n > v_n$ for all $n \in \mathbb{N}$. The sup distance between u and v is defined by

(1) $\qquad d_s(u, v) = \sup_n |u_n - v_n|,$

which induces the sup topology on the space X.

Let R be the social evaluation relation on X, viz., uRv for any pair u, $v \in X$ means that the infinite utility stream u is judged to be at least as good as another infinite utility stream v. For any fixed $u \in X$, define the lower contour set of $P(R)$ at $u \in X$ by

(2) $\qquad L_{P(R)}(u) = \{x \in X \mid uP(R)x\},$

where $P(R)$ denotes the asymmetric part of R, viz., for all u, $v \in X$, $uP(R)v$ holds if and only if uRv and $\sim vRu$. R is said to be *complete* if and only if, for all u, $v \in X$, uRv or vRu holds. R is said to be *transitive* if and only if, for all u, v, $w \in X$, uRv and vRw imply uRw. R is said to be an *ordering* if and only if it satisfies completeness as well as

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transitivity. Unlike most of the preceding work along the line of Koopmans and Diamond, where the social evaluation relation is assumed to be an ordering on X, this paper will invoke much weaker properties of R, which are defined as follows. For any $t \in \mathbb{N}$, a finite subset $\{u^1, u^2, ..., u^t\}$ of X is called a $P(R)$-cycle of order t if and only if $u^1 P(R) u^2$, $u^2 P(R) u^3$, ..., $u^t P(R) u^1$ hold. R is said to be *acyclic* if and only if there exists no $P(R)$-cycle of any order t, where $3 \le t < +\infty$. It is clear that the transitivity of R implies the acyclicity of R, and the converse implication does not hold in general. Concerning the continuity requirement on R, we will invoke the following semi-continuity axiom, which is weaker than Diamond's full continuity axiom.

Upper Semi-Continuity (USC)
　　　　For all $u \in X$, $L_{P(R)}(u)$ is an open set in X.

Unless otherwise stated, the underlying topology of X is assumed to be the sup topology induced by the sup distance function (1).
　　　　Another axiom which we will maintain throughout this paper is the following:

Strong Pareto Principle (SP)
　　　　For all $u, v \in X$, if $u > v$, then $u P(R) v$.

3. Pigou-Dalton Transfer Principle and Acyclic Social Evaluation

　　　　In contrast with the purely procedural equity principle à la Sidgwick and Diamond, we will introduce an axiom which embodies the consequentialist value in the form of preference for egalitarian distribution of utilities among generations. To be precise, our axiom reads as follows:

Pigou-Dalton Transfer Principle (PDT)
　　　　For any $u, v \in X$, if there exists a positive number $\varepsilon > 0$ and a pair $i, j \in \mathbb{N}$ such that

(3)　　　$v_i = u_i + \varepsilon \le u_j - \varepsilon = v_j$, $u_k = v_k$ for all $k \in \mathbb{N} - \{i, j\}$

holds, then $v P(R) u$ must be true.

　　　　Although the Pigou-Dalton Transfer Principle is substantially different from Diamond's

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purely procedural equity axiom, we cannot yet escape from the impossibility impasse in the presence of the continuity axiom (even in the much weaker form of upper semi-continuity), the Strong Pareto Principle, and the rationality postulate (even in the much weaker form of acyclicity).

Theorem 1

There exists no acyclic social evaluation relation R which satisfies the Upper Semi-Continuity (**USC**) with respect to the sup topology, and satisfies the Pigou-Dalton Transfer Principle (**PDT**) and the Strong Pareto Principle (**SP**).

Proof: Let $_{\text{con}}0$ be the infinite utility stream which repeats the utility level of 0 indefinitely. Start from $u^0 = (1, {}_{\text{con}}0)$ and define an infinite sequence of infinite utility streams $\{u^n \in X \mid n \in \mathbb{N}\}$ by

$$u^1 = (\frac{1}{2}, \frac{1}{2}, {}_{\text{con}}0); u^2 = (\frac{1}{4}, \frac{1}{2}, \frac{1}{4}, {}_{\text{con}}0); u^3 = (\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, {}_{\text{con}}0);$$

$$u^4 = (\frac{1}{8}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{8}, {}_{\text{con}}0); u^5 = (\frac{1}{8}, \frac{1}{8}, \frac{1}{4}, \frac{1}{4}, \frac{1}{8}, \frac{1}{8}, {}_{\text{con}}0);$$

$$u^6 = (\frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{4}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8}, {}_{\text{con}}0); u^7 = (\frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8}, {}_{\text{con}}0); \cdots .$$

By virtue of the axiom **PDT**, this infinite sequence satisfies

(4) $\quad u^n P(R) u^{n-1} P(R) \cdots P(R) u^1 P(R) u^0$

for all $n \in \mathbb{N}$, which implies that

(5) $\quad \sim u^0 P(R) u^n$

for all $n \in \mathbb{N}$ by virtue of the acyclicity of R. On the other hand, the axiom **SP** ensures that

(6) $\quad u^0 P(R) {}_{\text{con}}0.$

Observe that each and every term of the subsequence $\{u^{n^2-1} \in X \mid n \in \mathbb{N}\}$ of the original infi-

nite sequence $\{u^n \in X \mid n \in \mathbb{N}\}$ takes the form of

(7) $u^{n^2-1} = (\dfrac{1}{2^n}, \dfrac{1}{2^n}, \cdots, \dfrac{1}{2^n}, {}_{con}\mathbf{0})$ ($\dfrac{1}{2^n}$ is repeated 2^n times),

so that we are assured that

(8) $\lim\limits_{n \to \infty} d_s(u^{n^2-1}, {}_{con}\mathbf{0}) = \lim\limits_{n \to \infty} \dfrac{1}{2^n} = 0$

holds. By assumption, R is upper semi-continuous with respect to the sup norm. Thus, we are assured by (6) and (8) that

(9) $\exists\, n^* \in \mathbb{N},\ \forall\, n \in \mathbb{N}: n > n^* \Rightarrow u^0 P(R) u^{n^2-1}$

holds in contradiction with (5). ‖

4. Egalitarian Preference in Terms of the Lorenz Domination

For each $u \in X$ and each $n \in \mathbb{N}$, let ${}_1u_n$ be defined by ${}_1u_n = (u_1, u_2, \ldots, u_n)$. We may now introduce an alternative axiom of egalitarian preference in terms of the Lorenz domination between the truncated infinite utility streams as follows:

Lorenz Domination Principle (LD)

For any $u,\ v \in X$, if there exists $n^* \in \mathbb{N}$ such that (1) ${}_1u_{n^*}$ Lorenz dominates ${}_1v_{n^*}$, and (2) $u_n = v_n$ holds for all $n \in \mathbb{N}$ such that $n > n^*$, then $u P(R) v$ must be true.

Recollect that ${}_1u_{n^*}$ Lorenz dominates ${}_1v_{n^*}$ if and only if (i) $\sum\limits_{n=1}^{n^*} u_n = \sum\limits_{n=1}^{n^*} v_n$ and (ii) the

Lorenz curve of ${}_1u_{n^*}$ lies uniformly above the Lorenz curve of ${}_1v_{n^*}$. Recollect also that the Pigou-Dalton Transfer Principle and the Lorenz Domination Principle can be demonstrated to be equivalent if the social evaluation relation R is an ordering, but not necessarily otherwise. Nevertheless, the Lorenz Domination Principle can replace the Pigou-Dalton Transfer Principle

without vitiating the validity of the impossibility theorem even if we get rid of the rationality postulate in the form of completeness, transitivity or acyclicity of R altogether.

Theorem 2

There exists no social evaluation relation R which satisfies the Upper Semi-Continuity (**USC**) with respect to the sup topology, and satisfies the Lorenz Domination Principle (**LD**) and the Strong Pareto Principle (**SP**).

Proof: By virtue of the axiom **SP**, it is clear that

(10) $(1, {}_{con}0)\,P(R)\,{}_{con}0$

holds true. Invoking the axiom **USC** with respect to the sup topology, there exists $n^* \in \mathbb{N}$ such that

(11) $(1, {}_2 0_{2^{n*}}, {}_{con}0)P(R)\,(\dfrac{1}{2^{n*}}, \dfrac{1}{2^{n*}}, ..., \dfrac{1}{2^{n*}}, {}_{con}0)$ ($\dfrac{1}{2^{n*}}$ is repeated 2^{n^*} times),

whereas the axiom **LD** implies that

(12) $(\dfrac{1}{2^{n*}}, \dfrac{1}{2^{n*}}, ..., \dfrac{1}{2^{n*}}, {}_{con}0)P(R)\,(1, {}_2 0_{2^{n*}}, {}_{con}0)$ ($\dfrac{1}{2^{n*}}$ is repeated 2^{n^*} times).

It is clear that (11) and (12) are incompatible in view of the asymmetry of $P(R)$. ‖

As the first auxiliary step in generalizing **Theorem 2** in several respects, let us define l^∞ and l^∞_+ as the set of all bounded infinite sequences of real numbers and the set of all bounded infinite sequences of non-negative real numbers, respectively. In what follows, we assume that X is a non-empty subset of l^∞, which is comprehensive above, viz., for each $u \in X$ and $e \in l^\infty_+$, $u + e \in X$.

The second auxiliary step is the definition of the Weak Lorenz Domination Principle, which is given as follows.

Weak Lorenz Domination Principle (**WLD**)

For any $u, v \in X$, if there exists $n^* \in \mathbb{N}$ such that (1) ${}_1 u_{n*}$ Lorenz dominates ${}_1 v_{n*}$,

and (2) $u_n = v_n$ holds for all $n \in \mathbb{N}$ such that $n > n^*$, then $\sim v P(R) u$ must be true.

The third auxiliary step is the introduction of the locally solid linear topology on l^∞. For each $u \in X$, let $|u|$ be the infinite sequence of non-negative real numbers obtained from u by replacing each term u_n with the absolute value thereof. A subset A of X is said to be *solid* if, for all $u, v \in l^\infty$ with $|u| < |v|$ and $v \in A$, we have $u \in A$. A linear topology on l^∞ is said to be *locally solid* if it has a basis for $_{\text{con}}0$ consisting of solid sets. The sup topology is clearly locally solid. Note also that the locally solid linear topologies have been used in the literature on general equilibrium theory such as Andreu Mas-Collel (1986). See also Charalambos Aliprantis and Owen Burkinshaw (1978) for the characterizations and basic properties of the locally solid linear topologies.

The following simple fact is crucial for the generalization of **Theorem 2**.

Fact: For each $n \in \mathbb{N}$, let $u^n = (1/2^n, 1/2^n, \ldots, 1/2^n, {}_{\text{con}}0)$, where $1/2^n$ is repeated 2^n times. Then the sequence $\{u^n\}_{n=1}^\infty$ converges to $_{\text{con}}0$ with respect to any locally solid linear topology.

To verify this fact, let U be a set taken from the neighbourhood base of $_{\text{con}}0$ consisting of solid sets. Then $(1/2^n) \cdot {}_{\text{con}}1 \in U$ for large n, where $_{\text{con}}1$ is the infinite stream which repeats 1 indefinitely. Since U is solid, $u^n \in U$ for large n. This completes the proof.

Concerning the requirement of continuity of the social evaluation relation, we introduce the upper semi-continuity with respect to an arbitrary locally solid linear topology T on l^∞.

Upper Semi-Continuity with respect to T (**USC-T**)

For all $u \in X$, $L_{P(R)}(u)$ is an open set in X with respect to the topology T.

We may now announce the following:

Theorem 3

There exists no social evaluation relation R which satisfies the Upper Semi-Continuity with respect to the topology T (**USC-T**), and satisfies the Weak Lorenz Domination Principle (**WLD**) and the Strong Pareto Principle (**SP**).

Proof: Take any $v \in X$ and define $u = (1, {}_{\text{con}}0) + v$. X being comprehensive above, we have $u \in X$. By virtue of the axiom **SP**, we have $uP(R)v$. Let $u^n = (1/2^n, 1/2^n, \ldots,$

$1/2^n$, $_{\text{con}}0$) + v, where $1/2^n$ is repeated 2^n times. X being comprehensive above, we have u^n $\in X$ for all $n \in \mathbb{N}$. By virtue of the **Fact**, the sequence $\{u^n\}_{n=1}^{\infty}$ converges to v with respect to the topology T. By virtue of the axiom **USC-T**, $uP(R)u^n$ for some $n \in \mathbb{N}$. By virtue of the axiom **WLD**, $\sim uP(R)u^n$ for all $n \in \mathbb{N}$. It is clear that these two statements cannot both be true. $\|$

5. Concluding Remarks

Diamond's impossibility theorems, as well as most of the subsequent impossibility theorems, had a structure in common to the effect that the consequentialist axiom of the Strong Pareto Principle cannot but conflict with the purely procedural (hence non-consequentialist) axiom of Equity in the presence of the technical axiom either in the form of Continuity or in the form of Numerical Representability of the social evaluation relation. In sharp contrast, the impossibility theorems established in this paper differ from these prevailing results in that they show that the consequentialist axiom of the Strong Pareto Principle cannot but conflict with another consequentialist axiom of egalitarianism either in the form of the Pigou-Dalton Transfer Principle or in the form of the Lorenz Domination Principle in the presence of the weak version of Upper Semi-Continuity of the social evaluation relation. The message of our impossibility theorems seem to be rather different in this sense from the traditional im-possibility theorems. The purpose of this paper is served if we could bring this contrast into clear relief.

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