

Consequences, Opportunities, and Generalized Consequentialism and Non-consequentialism

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Abstract

In a recent paper to appear in *Journal of Economic Theory* [Kotaro Suzumura and Yongsheng Xu, “Characterizations of Consequentialism and Non-consequentialism”], an analytical framework was developed, which allowed us to characterize the concept of consequentialism and non-consequentialism. To simplify matters, however, the treatment in that paper was confined only to the cases where no active interactions exist between consequential considerations and procedural considerations. The present paper represents a generalization of our previous framework, which can now accommodate the situations where consequential considerations and procedural considerations actively interact. The analysis covers both the cases of finite and infinite number of alternatives. *Journal of Economic Literature* Classification Numbers: D00, D60, D63, D71.

Key Words: consequences, opportunities, extended preferences, consequentialism, non-consequentialism

1 Introduction

In recent years, people have gradually come to realize the importance of non-welfaristic features of the consequences in forming economic policy recommendations as well as in performing economic systems analysis. Some salient examples of non-welfaristic features of the consequences are individual and group rights, procedures through which consequences are realized, and opportunities from which outcomes are chosen.¹

In a recent paper, Suzumura and Xu (2000) developed several analytical frameworks which can accommodate situations where an individual expresses his extended preferences of the following type: it is better for him that an alternative x is brought about from the opportunity set A rather than another alternative y being brought about from the opportunity set B . An important feature of these frameworks is the novel concept of *extended alternatives* in the form of (x, A) with the intended interpretation of x being realized from the opportunity set A , which is used to evaluate alternative economic policies. Within these proposed frameworks, they defined various concepts of consequentialism and non-consequentialism and gave axiomatic characterizations of these concepts. The various concepts of consequentialism and non-consequentialism defined and characterized in Suzumura and Xu (2000) are the *extreme consequentialism*, *strong consequentialism*, *extreme non-consequentialism* and *strong non-consequentialism*.

However, their axiomatizations of consequentialism and non-consequentialism were concerned only with rather extreme cases where unequivocal priority was given to consequences (resp. opportunities) not only in the case of extreme consequentialism (resp. extreme non-consequentialism) but also in the case of strong consequentialism (resp. strong non-consequentialism). The purpose of this paper is to pursue a more general framework so that active interactions between consequential considerations and procedural considerations are allowed to play essential role. In other words, we would like to develop a framework which allows trade-off between the value of consequences and the richness of opportunities.

The structure of the paper is as follows. In Section 2, we present the basic notations and definitions. Section 3 discusses the generalized consequentialism and non-consequentialism in a simple framework in which the universal set is finite. In Section 4, we consider the generalized consequentialism and non-consequentialism in a context in which the universal set is infinite, but opportunity sets are finite, while in

Section 5, we consider the same issue in an economic environment where opportunity sets are compact subsets of the n -dimensional non-negative real space. Section 6 concludes the paper with some final remarks.

2 Basic Notations and Definitions

Let \mathcal{Z} and \mathcal{R} denote the set of all positive integers and the set of all real numbers, respectively. Let X , where $3 \leq \#X$, be the set of all mutually exclusive and jointly exhaustive *social states*. The elements of X will be denoted by x, y, z, \dots . K denotes a collection of subsets of X . The elements in K will be denoted by A, B, C, \dots , and they are called *opportunity sets*. Let $X \times K$ be the Cartesian product of X and K . Elements of $X \times K$ will be denoted by $(x, A), (y, B), (z, C), \dots$, and they are called *extended alternatives*. Let $\Omega = \{(x, A) | A \in K \text{ and } x \in A\}$. That is, Ω contains all (x, A) such that A is an element of K and x is an element of A . It should be clear that $\Omega \subseteq X \times K$, and for all $(x, A) \in \Omega, x \in A$ holds. For all $(x, A) \in \Omega$, the intended interpretation is that the alternative x is chosen from the opportunity set A .

Let \succeq be a reflexive, complete and transitive binary relation over Ω , viz., \succeq is an ordering over Ω . The asymmetric and symmetric parts of \succeq will be denoted by \succ and \sim , respectively. For any $(x, A), (y, B) \in \Omega$, $(x, A) \succeq (y, B)$ is interpreted as “choosing x from A is at least as good as choosing y from B .” Thus, in the extended framework, it is possible to give an expression to the intrinsic value of opportunity set in addition to the instrumental value thereof. Indeed, the decision-maker recognizes the intrinsic value of the opportunity of choice if there exists an extended alternative $(x, A) \in \Omega$ such that $(x, A) \succ (x, \{x\})$.

3 A Simple Framework

In this section, we confine our attention to the case where $\#X < \infty$, and K is the set of all non-empty subsets of X . We consider the following three axioms on the ordering \succeq , which are proposed by Suzumura and Xu (2000).

Independence (IND): For all $(x, A), (y, B) \in \Omega$, and all $z \in X - A \cup B$, $(x, A) \succeq (y, B) \Leftrightarrow (x, A \cup \{z\}) \succeq (y, B \cup \{z\})$.

Simple Indifference (SI): For all $x \in X$, and all $y, z \in X - \{x\}$, $(x, \{x, y\}) \sim (x, \{x, z\})$.

Simple Monotonicity (SM): For all $(x, A), (x, B) \in \Omega$, if $B \subset A$, then $(x, A) \succeq (x, B)$.

The following simple implication of (IND), (SI) and (SM) proves useful.

Lemma 3.1. Let \succeq be an ordering over Ω and satisfy (IND), (SI) and (SM). Then, for all $(a, A), (b, B) \in \Omega$, and all $x \in X - A, y \in X - B$, $(a, A) \succeq (b, B) \Leftrightarrow (a, A \cup \{x\}) \succeq (b, B \cup \{y\})$.

Proof. We first note that, by Theorem 3.1 of Suzumura and Xu (2000), the following is true:

Claim 3.2. For all $(a, A), (a, A') \in \Omega$, if $\#A = \#A'$, then $(a, A) \sim (a, A')$.

Let $(a, A), (b, B) \in \Omega, x \notin A, y \notin B$ and $(a, A) \succeq (b, B)$. Because \succeq is an ordering, we have only to show that $(a, A) \sim (b, B) \Rightarrow (a, A \cup \{x\}) \sim (b, B \cup \{y\})$ and $(a, A) \succ (b, B) \Rightarrow (a, A \cup \{x\}) \succ (b, B \cup \{y\})$.

First, we show that

$$(*) \quad (a, A) \sim (b, B) \Rightarrow (a, A \cup \{x\}) \sim (b, B \cup \{y\}).$$

Since $x \notin A$ and $y \notin B$, it is clear that $A \neq X$ and $B \neq X$. We consider several cases: (i) $A = \{a\}$; (ii) $B = \{b\}$; and (iii) $\#A > 1$ and $\#B > 1$.

(i) $A = \{a\}$. In this case, we distinguish two sub-cases: (i.1) $x \notin B$; and (i.2) $x \in B$. Consider (i.1). Since $x \notin B$, it follows from $(a, \{a\}) \sim (b, B)$ and (IND) that $(a, \{a, x\}) \sim (b, B \cup \{x\})$. By Claim 3.2, $(b, B \cup \{x\}) \sim (b, B \cup \{y\})$. Transitivity of \succeq now implies that $(a, \{a, x\}) \sim (b, B \cup \{y\})$. Now, consider (i.2) in which $x \in B$. To begin with, consider the sub-case where $B \cup \{y\} = \{a, b\}$. Given that $x \notin A$ and $y \notin B$, we have $x = b$ and $y = a$. Since $\#X \geq 3$, there exists $c \in X$ such that $c \notin \{a, b\}$. It follows from $(a, \{a\}) \sim (b, \{b\}) = (b, B)$ and (IND) that $(a, \{a, c\}) \sim (b, \{b, c\})$. From Claim 3.2, $(a, \{a, b\}) \sim (a, \{a, c\})$, and $(b, \{b, c\}) \sim (b, \{a, b\})$. Then, transitivity of \succeq implies $(a, \{a, b\}) \sim (b, \{a, b\})$; that is, $(a, \{a, x\}) \sim (b, B \cup \{y\})$. Turn now to the sub-case where $B \cup \{y\} \neq \{a, b\}$. If $y \neq a$, starting with $(a, \{a\}) \sim (b, B)$, by (IND), $(a, \{a, y\}) \sim (b, B \cup \{y\})$. By Claim 3.2, $(a, \{a, x\}) \sim (a, \{a, y\})$. Transitivity of \succeq implies that $(a, \{a, x\}) \sim (b, B \cup \{y\})$. If $y = a$, given that $\#X \geq 3$ and $B \cup \{y\} \neq \{a, b\}$, there exists $z \in B$ such that $z \notin \{a, b\}$. By Claim 3.2, $(b, B) \sim (b, (B \cup \{y\}) -$

$\{z\}$). From $(a, \{a\}) \sim (b, B)$, transitivity of \succeq implies $(a, \{a\}) \sim (b, (B \cup \{y\}) - \{z\})$. Now, noting that $z \neq a$, by (IND), $(a, \{a, z\}) \sim (b, B \cup \{z\})$ holds. From Claim 3.2, $(a, \{a, x\}) \sim (a, \{a, z\})$. Transitivity of \succeq now implies $(a, \{a, x\}) \sim (b, B \cup \{y\})$.

(ii) $B = \{b\}$. This case can be dealt similarly as case (i).

(iii) $\#A > 1$ and $\#B > 1$. Consider $A', A'' \in K$ such that $\{a, b\} \subseteq A'' \subseteq A'$, $\#A'' = \min(\#A, \#B) > 1$, $\#A' = \max(\#A, \#B) > 1$. Since $A \neq X$ and $B \neq X$, the existence of such A' and A'' is guaranteed. It should be clear that there exists $z \in X$ such that $z \notin A'$. If $\#A \geq \#B$, consider (a, A') and (b, A'') . From Claim 3.2, $(a, A') \sim (b, A'')$ follows from the construction of A' and A'' , the assumption that $(a, A) \sim (b, B)$, and transitivity of \succeq . Note that there exists $z \in X - A'$. By (IND), $(a, A' \cup \{z\}) \sim (b, A'' \cup \{z\})$. By virtue of Claim 3.2, noting that $\#(A \cup \{x\}) = \#(A' \cup \{z\})$ and $\#(B \cup \{y\}) = \#(A'' \cup \{z\})$, $(a, A \cup \{x\}) \sim (b, B \cup \{y\})$ follows easily from transitivity of \succeq . If $\#A < \#B$, consider (a, A'') and (b, A') . Following a similar argument as above, we can show that $(a, A \cup \{x\}) \sim (b, B \cup \{y\})$. Thus, (*) is proved. The next order of our business is to show that

$$(**) \quad (a, A) \succ (b, B) \Rightarrow (a, A \cup \{x\}) \succ (b, B \cup \{y\}).$$

We distinguish several cases: (a) $A = \{a\}$; (b) $B = \{b\}$; and (c) $\#A > 1$ and $\#B > 1$.

(a) $A = \{a\}$. (a.1) $x \notin B$. In this sub-case, from $(a, \{a\}) \succ (b, B)$, by (IND), we obtain $(a, \{a, x\}) \succ (b, B \cup \{x\})$. By Claim 3.2, $(b, B \cup \{x\}) \sim (b, B \cup \{y\})$. Transitivity of \succeq implies that $(a, \{a, x\}) \succ (b, B \cup \{y\})$. (a.2) $x \in B$. If $B \cup \{y\} = \{a, b\}$, then, given that $x \notin A$ and $y \notin B$, we have $x = b$ and $y = a$. Since $\#X \geq 3$, there exists $c \in X$ such that $c \notin \{a, b\}$. It follows from $(a, \{a\}) \succ (b, \{b\}) = (b, B)$ and (IND) that $(a, \{a, c\}) \succ (b, \{b, c\})$. From Claim 3.2, $(a, \{a, b\}) \sim (a, \{a, c\})$ and $(b, \{b, c\}) \sim (b, \{a, b\})$. Transitivity of \succeq implies $(a, \{a, b\}) \succ (b, \{a, b\})$; i.e., $(a, \{a, x\}) \succ (b, B \cup \{y\})$. If $B \cup \{y\} \neq \{a, b\}$, we consider (a.2.i) $y \neq a$ and (a.2.ii) $y = a$. Suppose that (a.2.i) $y \neq a$. From $(a, \{a\}) \succ (b, B)$, by (IND), $(a, \{a, y\}) \succ (b, B \cup \{y\})$. By Claim 3.2, $(a, \{a, x\}) \sim (a, \{a, y\})$. Transitivity of \succeq now implies $(a, \{a, x\}) \succ (b, B \cup \{y\})$. Suppose next that (a.2.ii) $y = a$. Since $\#X \geq 3$ and $B \cup \{y\} \neq \{a, b\}$, there exists $c \in B$ such that $c \notin \{a, b\}$. By Claim 3.2, $(b, B) \sim (b, (B \cup \{y\}) - \{c\})$. From $(a, \{a\}) \succ (b, B)$, by transitivity of \succeq , $(a, \{a\}) \succ (b, (B \cup \{y\}) - \{c\})$. Now, noting that $c \neq a$, by (IND), $(a, \{a, c\}) \succ (b, B \cup \{y\})$. From Claim 3.2, $(a, \{a, c\}) \sim (a, \{a, x\})$. Transitivity of \succeq implies $(a, \{a, x\}) \succ (b, B \cup \{y\})$.

(b) $B = \{b\}$. If $x \notin B$, it follows from $(a, A) \succ (b, B)$ and (IND) that $(a, A \cup \{x\}) \succ (b, \{b, x\})$. By Claim 3.2, $(b, \{b, x\}) \sim (b, \{b, y\})$. Transitivity of \succeq now implies $(a, A \cup \{x\}) \succ (b, \{b, y\})$. If $x \in B$, then $x = b$. Consider first the case where $y = a$. If $A = \{a\}$, it follows from (a) that $(a, \{a, x\}) \succ (b, \{b, y\})$. Suppose $A \neq \{a\}$. Given that $x = b$, $y = a$, $x \notin A$, and $y \notin B$, and noting that $\#X \geq 3$, there exists $c \in A - \{a, b\}$. From Claim 3.2, $(a, (A \cup \{x\}) - \{c\}) \sim (a, A)$. From transitivity of \succeq and noting that $(a, A) \succ (b, \{b\})$, $(a, (A \cup \{x\}) - \{c\}) \succ (b, \{b\})$ holds. By (IND), $(a, A \cup \{x\}) \succ (b, \{b, c\})$. From Claim 3.2, $(b, \{b, y\}) \sim (b, \{b, c\})$. Therefore, $(a, A \cup \{x\}) \succ (b, \{b, y\})$ follows easily from transitivity of \succeq . Consider next that $y \neq a$. If $y \notin A$, then, by (IND) and $(a, A) \succ (b, \{b\})$, we obtain $(a, A \cup \{y\}) \succ (b, \{b, y\})$ immediately. By Claim 3.2, $(a, A \cup \{y\}) \sim (a, A \cup \{x\})$. Transitivity of \succeq implies $(a, A \cup \{x\}) \succ (b, \{b, y\})$. If $y \in A$, noting that $y \neq a$, $y \notin B$, and $x = b$, we have $\#((A \cup \{x\}) - \{y\}) = \#A$. By Claim 3.2, $(a, A) \sim (a, (A \cup \{x\}) - \{y\})$. Transitivity of \succeq implies $(a, (A \cup \{x\}) - \{y\}) \succ (b, \{b\})$. By (IND), it then follows that $(a, A \cup \{x\}) \succ (b, \{b, y\})$.

(c) $\#A > 1$ and $\#B > 1$. This case is similar to the case (iii) above, and we may safely omit it.

Thus, (**) is proved. (*) together with (**) completes the proof of Lemma 3.1.

■

We are now ready to characterize completely the class of all orderings which satisfy (IND), (SI) and (SM).

Theorem 3.3. \succeq satisfies (IND), (SI) and (SM) if and only if there exist a function $u : X \rightarrow \mathcal{R}$ and a function $f : \mathcal{R} \times \mathcal{Z} \rightarrow \mathcal{R}$ such that

$$(3.1) \text{ For all } x, y \in X, u(x) \geq u(y) \Leftrightarrow (x, \{x\}) \succeq (y, \{y\});$$

$$(3.2) \text{ For all } (x, A), (y, B) \in \Omega, (x, A) \succeq (y, B) \Leftrightarrow f(u(x), \#A) \geq f(u(y), \#B);$$

(3.3) f is non-decreasing in each of its arguments and has the following property:

For all integers $i, j, k \geq 1$ and all $x, y \in X$, if $i + k, j + k \leq \#X$, then

$$(3.3.1) f(u(x), i) \geq f(u(y), j) \Leftrightarrow f(u(x), i + k) \geq f(u(y), j + k)$$

Proof. We first show that if \succeq satisfies (IND), (SI) and (SM), then there exist a function $f : \mathcal{R} \times \mathcal{Z} \rightarrow \mathcal{R}$ and a function $u : X \rightarrow \mathcal{R}$ such that (3.1), (3.2) and (3.3)

of Theorem 3.3 hold.² Let \succeq satisfy (IND), (SI) and (SM). Since \succeq is an ordering, X is finite and so is Ω as well, there exist $u : X \rightarrow \mathcal{R}$ and $F : \Omega \rightarrow \mathcal{R}$ such that

$$(3.4) \text{ For all } x, y \in X, (x, \{x\}) \succeq (y, \{y\}) \Leftrightarrow u(x) \geq u(y);$$

$$(3.5) \text{ For all } (x, A), (y, B) \in \Omega, (x, A) \succeq (y, B) \Leftrightarrow F(x, A) \geq F(y, B).$$

Clearly, (3.1) of Theorem 3.3 is satisfied. We now show that, for all $(x, A), (y, B) \in \Omega$, if $u(x) = u(y)$ and $\#A = \#B$, then $(x, A) \sim (y, B)$. Let $(x, A), (y, B) \in \Omega$ be such that $u(x) = u(y)$ and $\#A = \#B$. From $u(x) = u(y)$, we must have $(x, \{x\}) \sim (y, \{y\})$. Then, if necessary, by the repeated use of Lemma 3.1 and noting that $\#A = \#B$, $(x, A) \sim (y, B)$ can be obtained easily. Now, define $\Sigma \subset \mathcal{R} \times \mathcal{Z}$ as follows: $\Sigma := \{(t, i) \in \mathcal{R} \times \mathcal{Z} \mid \exists (x, A) \in \Omega \text{ such that } t = u(x) \text{ and } i = \#A\}$. Next, define the binary relation \geq^* on Σ as follows: For all $(x, A), (y, B) \in \Omega$, $(x, A) \succeq (y, B) \Leftrightarrow (u(x), \#A) \geq^* (u(y), \#B)$. From the above discussion and noting that \succeq satisfies (SM) and (IND), the binary relation \geq^* defined on Σ is an ordering and has the following properties:

(SM'): For all $(t, i), (t, j) \in \Sigma$, if $j \geq i$ then $(t, j) \geq^* (t, i)$;

(IND'): For all $(s, i), (t, j) \in \Sigma$, and all integer k , if $i + k \leq \#X$ and $j + k \leq \#X$, then $(s, i) \geq^* (t, j) \Leftrightarrow (s, i + k) \geq^* (t, j + k)$.

Since Σ is finite and \geq^* is an ordering on Σ , there exists a function $f : \mathcal{R} \times \mathcal{Z} \rightarrow \mathcal{R}$ such that, for all $(s, i), (t, j) \in \Sigma$, $(s, i) \geq^* (t, j)$ iff $f(s, i) \geq f(t, j)$. From the definition of \geq^* and Σ , we must have the following: For all $(x, A), (y, B) \in \Omega$, $(x, A) \succeq (y, B) \Leftrightarrow (u(x), \#A) \geq^* (u(y), \#B) \Leftrightarrow f(u(x), \#A) \geq f(u(y), \#B)$. To prove that f is non-decreasing in each of its arguments, we first consider the case in which $u(x) \geq u(y)$ and $\#A = \#B$. Given $u(x) \geq u(y)$, from the definition of u , we have $(x, \{x\}) \succeq (y, \{y\})$. Noting that $\#A = \#B$, by the repeated use of Lemma 3.1, if necessary, we must have $(x, A) \succeq (y, B)$. Thus, f is non-decreasing in its first argument. To show that f is non-decreasing in its second argument, we consider the case in which $u(x) = u(y)$ and $\#A \geq \#B$. From $u(x) = u(y)$, we must have $(x, \{x\}) \sim (y, \{y\})$. Then, from the earlier argument, $(x, A') \sim (y, B)$ where $A' \subset A$ is such that $\#A' = \#B$. Now, by (SM), $(x, A) \succeq (x, A')$. Then, $(x, A) \succeq (y, B)$ follows from the transitivity of \succeq . Therefore, f is non-decreasing in each of its arguments. Finally, (3.3.1) follows easily from (IND').

To check the necessity part of the theorem, suppose $u : X \rightarrow \mathcal{R}$ and $f : \mathcal{R} \times \mathcal{Z} \rightarrow \mathcal{R}$ are such that (3.1), (3.2) and (3.3) of Theorem 3.3 are satisfied.

(SI): For all $x \in X$ and all $y, z \in X - \{x\}$, we note that $\#\{x, y\} = \#\{x, z\}$, therefore, $f(u(x), \#\{x, y\}) = f(u(x), \#\{x, z\})$, which in turn implies that $(x, \{x, y\}) \sim (x, \{x, z\})$ holds.

(SM): For all $(x, A), (x, B) \in \Omega$ such that $B \subseteq A$. Then, $f(u(x), \#A) \geq f(u(x), \#B)$ follows from the fact that f is non-decreasing in each of its arguments and $\#A \geq \#B$. Therefore, $(x, A) \succeq (x, B)$.

(IND): For all $(x, A), (y, B) \in \Omega$, and all $z \in X - A \cup B$, from (3.3.1), $f(u(x), \#A) \geq f(u(y), \#B) \Leftrightarrow f(u(x), \#A + 1) \geq f(u(y), \#B + 1) \Leftrightarrow f(u(x), \#(A \cup \{z\})) \geq f(u(y), \#(B \cup \{z\}))$. Therefore, $(x, A) \succeq (y, B) \Leftrightarrow (x, A \cup \{z\}) \succeq (y, B \cup \{z\})$.

■

Remark 3.4. Although Theorem 3.3 allows us to treat all the cases where the utility of consequential states and the value of richness of opportunities actively interact, it also covers the polar cases of consequentialism and non-consequentialism defined in Suzumura and Xu (2000). Indeed, they can be obtained as special cases of Theorem 3.3 as follows:

Extreme Consequentialism: For all $(x, A) \in \Omega$, $f(u(x), \#A) = u(x)$.

Strong Consequentialism: For all $(x, A), (y, B) \in \Omega$, $(x, A) \succeq (y, B) \Leftrightarrow [u(x) > u(y) \text{ or } (u(x) = u(y) \text{ and } \#A \geq \#B)]$, where $u : X \rightarrow \mathcal{R}$ is such that for all $x, y \in X$, $u(x) \geq u(y) \Leftrightarrow (x, \{x\}) \succeq (y, \{y\})$.

Extreme Non-consequentialism: For all $(x, A) \in \Omega$, $f(u(x), \#A) = \#A$.

Strong Non-consequentialism: For all $(x, A), (y, B) \in \Omega$, $(x, A) \succeq (y, B) \Leftrightarrow [\#A > \#B \text{ or } (\#A = \#B \text{ and } u(x) \geq u(y))]$, where $u : X \rightarrow \mathcal{R}$ is such that for all $x, y \in X$, $u(x) \geq u(y) \Leftrightarrow (x, \{x\}) \succeq (y, \{y\})$.

4 Finite Opportunity Sets

Although Theorem 3.3 provides us with a full characterization result for all orderings satisfying (IND), (SI) and (SM), it hinges on the restrictive assumption to the effect that X is finite. In this section, we discuss the case in which X contains an infinite number of alternatives, but K consists of the set of all finite non-empty subsets of X . Consequently, Ω contains all (x, A) such that A is finite and x is an element of A .

In this arena, the necessary and sufficient condition for an ordering over Ω to satisfy (IND), (SI) and (SM) reads as follows.

Theorem 4.1. \succ satisfies (IND), (SI) and (SM) if and only if there exists an ordering $\succeq^\#$ on $X \times \mathcal{Z}$ such that

$$(4.1) \text{ For all } (x, A), (y, B) \in \Omega, (x, A) \succeq (y, B) \Leftrightarrow (x, \#A) \succeq^\# (y, \#B);$$

$$(4.2) \text{ For all integers } i, j, k \geq 1 \text{ and all } x, y \in X, (x, i) \succeq^\# (y, j) \Leftrightarrow (x, i+k) \succeq^\# (y, j+k); \text{ and } (x, i+k) \succeq^\# (x, i).$$

Proof. First, we note that Claim 3.2 holds; that is, for all $(a, A), (a, B) \in \Omega$, if $\#A = \#B$, then $(a, A) \sim (a, B)$. Next, we show the following:

Claim 4.2. For all $x, y \in X$ and all $(x, A), (y, A) \in \Omega$, $(x, \{x\}) \succeq (y, \{y\}) \Leftrightarrow (x, A) \succeq (y, A)$.

Let $x, y \in X$ and $(x, A), (y, A) \in \Omega$. If $x = y$, then Claim 4.2 follows from the reflexivity of \succeq immediately. Let $x \neq y$. Suppose that $(x, \{x\}) \sim (y, \{y\})$. Let $z \in X - \{x, y\}$. By (IND), we have $(x, \{x, z\}) \sim (y, \{y, z\})$. From Claim 3.2, we must have $(x, \{x, z\}) \sim (x, \{x, y\})$ and $(y, \{y, z\}) \sim (y, \{x, y\})$. Then, by transitivity of \succeq , we obtain $(x, \{x, y\}) \sim (y, \{x, y\})$. Noting that A is finite, by the repeated use of (IND), we have $(x, A) \sim (y, A)$. Similarly, we can show that if $(x, \{x\}) \succ (y, \{y\})$ then $(x, A) \succ (y, A)$. Since \succeq is an ordering, Claim 4.2 is proved.

We now show that, for all $(x, A), (y, B) \in \Omega$, if $(x, \{x\}) \sim (y, \{y\})$ and $\#A = \#B$, then $(x, A) \sim (y, B)$. Let $C \in K$ be such that $\#C = \#A = \#B$ and $\{x, y\} \subseteq C$. From Claim 4.2, we have $(x, C) \sim (y, C)$. Note that $(x, C) \sim (x, A)$ and $(y, C) \sim (y, B)$ follow from Claim 3.2. By transitivity of \succeq , we have $(x, A) \sim (y, B)$. Define the binary relation $\succeq^\#$ on $X \times \mathcal{Z}$ as follows: For all $x, y \in X$ and all positive integers i, j , $(x, i) \succeq^\# (y, j) \Leftrightarrow [(x, A) \succeq (y, B) \text{ for some } A, B \in K \text{ such that } x \in A, y \in B, i = \#A, j = \#B]$. From the above discussion, $\succeq^\#$ is well-defined and is an ordering. A similar method of proving (3.3) can be invoked to prove that (4.2) holds. ■

Remark 4.3. Based on Theorem 4.1, the concepts of extreme consequentialism, strong consequentialism, extreme non-consequentialism and strong consequentialism defined in Suzumura and Xu (2000) can be expressed similarly as in the last section.

Remark 4.4. It should be clear that without imposing any other condition on

the extended preference ordering \succeq , \succeq may not be representable by any numerical function. In order for \succeq to be numerically representable, further restrictions on \succeq must be imposed. In the next, we present a small step toward the numerical representation of the extended preference ordering \succeq over Ω . Assume therefore that $X = \mathcal{R}_+^n$ for the rest of this section, and define the following property:

Continuity (CON): For all $x^i \in X (i = 1, 2, \dots)$ and all $x, y \in X$, if $\lim_{i \rightarrow \infty} x^i = x$, then $[(x^i, \{x^i\}) \succeq (y, \{y\}) \text{ for } i = 1, 2, \dots] \Rightarrow (x, \{x\}) \succeq (y, \{y\})$, and $[(y, \{y\}) \succeq (x^i, \{x^i\}) \text{ for } i = 1, 2, \dots] \Rightarrow (y, \{y\}) \succeq (x, \{x\})$.

Then we may assert the following proposition:

Theorem 4.5. Suppose that $X = \mathcal{R}_+^n$. Suppose also that \succeq satisfies (IND), (SI), (SM) and (CON). Then there exist a continuous function $u : X \rightarrow \mathcal{R}$ and an ordering \succeq^* on $\mathcal{R} \times \mathcal{Z}$ such that

- (a) For all $(x, A), (y, B) \in \Omega$, $(x, A) \succeq (y, B) \Leftrightarrow (u(x), \#A) \succeq^* (u(y), \#B)$;
- (b) For all $x, y \in X$, $(x, \{x\}) \succeq (y, \{y\}) \Leftrightarrow u(x) \geq u(y)$;
- (c) For all integers $i, j, k \geq 1$, and all $x, y \in X$, $(u(x), i) \succeq^* (u(y), j) \Leftrightarrow (u(x), i + k) \succeq^* (u(y), j + k)$; and $(u(x), i + k) \succeq^* (u(x), i)$.

Proof. From Theorem 4.1, we know that there exists an ordering $\succeq^\#$ on $X \times \mathcal{Z}$ such that

- (F.1) For all $(x, A), (y, B) \in \Omega$, $(x, A) \succeq (y, B) \Leftrightarrow (x, \#A) \succeq^\# (y, \#B)$;
- (F.2) For all integers $i, j, k \geq 1$ and all $x, y \in X$, $(x, i) \succeq^\# (y, j) \Leftrightarrow (x, i + k) \succeq^\# (y, j + k)$; and $(x, i + k) \succeq^\# (x, i)$.

Now, for all $x, y \in X$, define the binary relation R on X as follows: For all $x, y \in X$, $xRy \Leftrightarrow (x, \{x\}) \succeq (y, \{y\})$. Since \succeq is an ordering and satisfies (CON), R on $X = \mathcal{R}_+^n$ is an ordering and satisfies the following continuity property: For all $x \in X$, $\{y \in X | yRx\}$ and $\{y \in X | xRy\}$ are closed. Therefore, there exists a continuous function $u : X \rightarrow \mathcal{R}$ such that for all $x, y \in X$, $xRy \Leftrightarrow u(x) \geq u(y)$. From the definition of R , it is clear that for all $x, y \in X$, $(x, \{x\}) \succeq (y, \{y\}) \Leftrightarrow u(x) \geq u(y)$. Therefore, (b) of Theorem 4.5 holds. Given that (b) holds, it is straightforward to check that (a) and (c) hold as well. ■

5 Economic Environment

Although Theorem 4.5 is somewhat reassuring, it is still very limiting if we are confined to the finite opportunity sets. Assume therefore that $X = \mathcal{R}_+^n$, and let K be the family of all compact subsets of X . For any $A \subset X$, let $\beta(A)$ be the *closure* of A . For technical reasons, we now expand Ω to Ω^* , where $\Omega^* := \Omega \cup \{(x, \emptyset) | x \in X\}$. Likewise, we also expand the domain of the extended preference \succeq to Ω^* by defining $(x, A) \succeq (x, \emptyset)$ for all $(x, A) \in \Omega$. The intended interpretation of this extension is that (x, \emptyset) represents an empty promise of realizing a state x without actually giving any real opportunity of choosing x , which is never preferred to whatever $(x, A) \in \Omega$ we may care to specify. Let $\Omega^1 = \{(x, \{x\}) | x \in X\}$.

We consider the following axioms for the ordering \succeq over Ω .

Denseness (DEN): For all $(x, A), (x, B) \in \Omega^*$ such that $(x, A) \succ (x, B)$, there exists $(x, A') \in \Omega^*$ such that $A' \subseteq A$, $A' \neq \emptyset$ and $(x, A') \sim (x, B)$.

Archimedean Property (ARP): For all $(x, A), (x, B) \in \Omega^*$, if $(x, A) \succ (x, B) \succ (x, \emptyset)$, then there exist $(x, C_1), \dots, (x, C_m) \in \Omega^*$ for some $m \in \mathcal{Z}$ such that $(x, C_1) \sim \dots \sim (x, C_m) \sim (x, B)$ and $(x, B \cup C_1 \cup \dots \cup C_m) \succeq (x, A)$.

Composition (COM): For all $A, B, C \subset X$ and all $x, y \in X$, if $A \cap C = B \cap C = \emptyset$, $(x, \beta(A)), (y, \beta(B)), (x, \beta(A \cup C)), (y, \beta(B \cup C)) \in \Omega^*$, then $(x, \beta(A)) \succeq (y, \beta(B))$ iff $(x, \beta(A \cup C)) \succeq (y, \beta(B \cup C))$.

Separability (SEP): For all $(x, A), (x, B), (y, A), (y, B) \in \Omega^*$, $(x, A) \succeq (x, B) \Leftrightarrow (y, A) \succeq (y, B)$.

Similar versions of Denseness, Archimedean Property and Composition were proposed by Pattanaik and Xu (2000) in the context of ranking compact opportunity sets in terms of freedom of choice.

Theorem 5.1. If \succeq is an ordering and satisfies (DEN), (ARP), (COM) and (SEP), then there exist a function $\sigma : K \rightarrow \mathcal{R}$ and an ordering \geq^* on $\Omega^1 \times \mathcal{R}$ such that

- (5.1) σ is countably additive; $\sigma(\emptyset) = 0$; For all $A, B \in K$, if $B \subset A$ then $\sigma(A) \geq \sigma(B)$; For all $A, B \in K$ and all $x \in X$, $(x, A \cup \{x\}) \succeq (x, B \cup \{x\}) \Leftrightarrow \sigma(A) \geq \sigma(B)$;

(5.2) For all $(a, A), (b, B) \in \Omega^*$, $(a, A) \succeq (b, B) \Leftrightarrow ((a, \{a\}), \sigma(A)) \geq^* ((b, \{b\}), \sigma(B))$.

Proof. For all $A, B \in K$ and all $x \in X$, define the binary relation \succeq_x on K as follows:

$$A \succeq_x B \Leftrightarrow (x, A \cup \{x\}) \succ (x, B \cup \{x\}).$$

Clearly, \succeq_x is an ordering. Let \succ_x and \sim_x be the asymmetric part and symmetric part of \succeq_x , respectively. We now show that \succeq_x satisfies the following properties:

P.1. For all $A, B \in K$, if $A \succ_x B$, then there exists $A' \in K$ such that $A' \subseteq A$, $A' \neq \emptyset$ and $A' \sim B$.

P.2. For all $A, B \in K$, if $A \succ_x B \succ_x \emptyset$, then there exist $C_1, \dots, C_m \in K$ for some $m \in \mathcal{Z}$ such that $C_1 \sim_x \dots \sim_x C_m$ and $B \cup C_1 \cup \dots \cup C_m \succeq_x A$.

P.3. For all $A, B, C \subseteq X$, if $\beta(A), \beta(B), \beta(C) \in K$, and $A \cap C = B \cap C = \emptyset$, then $\beta(A) \succeq_x \beta(B) \Leftrightarrow \beta(A \cup C) \succeq_x \beta(B \cup C)$.

P.4. For all $x, y \in X$, $\succeq_x = \succeq_y$.

To check that \succeq_x satisfies P.1, let $A, B \in K$ be such that $A \succ_x B$. From the definition of \succeq_x , we must have $(x, A \cup \{x\}) \succ (x, B \cup \{x\})$. Then, from (DEN), there exists $(x, A') \in \Omega^*$ such that $A' \subseteq A \cup \{x\}$, $A' \neq \emptyset$, and $(x, A') \sim (x, B \cup \{x\})$. Note that $x \in A'$. Therefore, $A' \sim_x B$. P.2 and P.3 can be checked similarly. Finally, P.4 follows from (SEP) easily.

Because of P.4, we can define $\succeq^* = \succeq_x$ for all $x \in X$ without ambiguity. Then, noting that \succeq^* satisfies P.1, P.2 and P.3, by Theorem 4.2 of Pattanaik and Xu (2000), there exists a function $\sigma : K \rightarrow \mathcal{R}$ such that σ is countably additive, $\sigma(\emptyset) = 0$, and for all $A, B \in K$, $B \subseteq A \Rightarrow \sigma(A) \geq \sigma(B)$, and for all $A, B \in K$, $A \succeq^* B \Leftrightarrow \sigma(A) \geq \sigma(B)$; thus (5.1) holds.

From the definition of \succeq^* and the property of σ function, clearly, we have the following:

P.5. For all $(x, A), (x, B) \in \Omega^*$, if $\sigma(A) = \sigma(B)$, then $(x, A) \sim (x, B)$.

We also note that

P.6. For all $x, y \in X$, $(x, \{x\}) \sim (x, \emptyset)$.

To see that P.6 is true, suppose to the contrary that $\text{not}[(x, \{x\}) \sim (x, \emptyset)]$. By

assumption that \succeq is an ordering and $(x, \{x\}) \succeq (x, \emptyset)$, we then must have $(x, \{x\}) \succ (x, \emptyset)$. Then, by (DEN), there must exist $(x, A') \in \Omega^*$ such that $A' \neq \emptyset$, $A' \subseteq \{x\}$ and $(x, A') \sim (x, \emptyset)$, which is a contradiction. Therefore, $(x, \{x\}) \sim (x, \emptyset)$.

Thus, $\sigma(\{x\}) = 0$ for all $x \in X$. Since σ is countably additive, clearly, $\sigma(\{x, y\}) = 0$ for all $x, y \in X$. By P.5 and P.6, the following follows easily:

P.7. For all $x, y \in X$, $(x, \{x, y\}) \sim (x, \{x\}) \sim (x, \emptyset)$.

With P.7, the following can be shown to be true:

P.8. For all $x \in X$ and all $(a, A) \in \Omega^*$, $(a, A) \sim (a, A \cup \{x\})$.

To see that P.8 holds, we consider two cases: $x \in A$ and $x \notin A$. If $x \in A$, by virtue of the reflexivity of \succeq , P.8 holds. If $x \notin A$, we consider $(a, \{a, x\})$, $(a, \{a\})$ and $C = A - \{a\}$. From P.7, $(a, \{a, x\}) \sim (a, \{a\})$. Then, by (COM), we obtain $(a, \{a, x\} \cup C) = (a, A \cup \{x\}) \sim (a, \{a\} \cup A) = (a, A)$ immediately.

To show (5.2), we must show first that, for all $(a, A), (b, B) \in \Omega^*$, if $(a, \{a\}) \sim (b, \{b\})$ and $\sigma(A) = \sigma(B)$, then $(a, A) \sim (b, B)$. Let $(a, A), (b, B) \in \Omega^*$ be such that $(a, \{a\}) \sim (b, \{b\})$ and $\sigma(A) = \sigma(B)$. It should be clear that $a \in A$. Let $C = A - \{a, b\}$. Then, by (COM), from $(a, \{a\}) \sim (b, \{b\})$, we obtain $(a, \beta(\{a\} \cup C)) \sim (b, \beta(\{b\} \cup C))$. From P.8, $(a, \beta(\{a\} \cup C) \cup \{b\}) = (a, A \cup \{b\}) \sim (a, \beta(\{a\} \cup C))$, and $(b, \beta(\{b\} \cup C) \cup \{a\}) = (b, A \cup \{b\}) \sim (b, \beta(\{b\} \cup C))$. Therefore, by the transitivity of \succeq , we obtain $(a, A \cup \{b\}) \sim (b, A \cup \{b\})$. Note, however, $\sigma(A \cup \{b\}) = \sigma(A) = \sigma(B)$. From P.5, we then have $(a, A) \sim (a, A \cup \{b\})$ and $(b, A \cup \{b\}) \sim (b, B)$. Hence, $(a, A) \sim (b, B)$ follows from the transitivity of \succeq immediately. Therefore, for all $(x, A), (y, B) \in \Omega^*$, in order to rank them according to \succeq , all the relevant information needed are $((x, \{x\}), \sigma(A))$ and $((y, \{y\}), \sigma(B))$. In other words, $(x, A) \succeq (y, B) \Leftrightarrow ((x, \{x\}), \sigma(A)) \geq^* ((y, \{y\}), \sigma(B))$ for some ordering \geq^* on $\Omega^1 \times \mathcal{R}$. ■

Remark 5.2. According to Theorem 5.1, we can now define the extreme consequentialism, strong consequentialism, extreme non-consequentialism and strong non-consequentialism as follows:

Extreme Consequentialism: For all $(a, A), (b, B) \in \Omega^*$, $(a, A) \succeq (b, B) \Leftrightarrow (a, \{a\}) \succeq (b, \{b\})$.

Strong Consequentialism: For all $(a, A), (b, B) \in \Omega^*$, $(a, A) \succeq (b, B) \Leftrightarrow [(a, \{a\}) \succ (b, \{b\})]$

$(b, \{b\})$, or $((a, \{a\}) \sim (b, \{b\})$ and $\sigma(A) \geq \sigma(B))$].

Extreme Non-consequentialism: For all $(a, A), (b, B) \in \Omega^*$, $(a, A) \succeq (b, B) \Leftrightarrow \sigma(A) \geq \sigma(B)$.

Strong Non-consequentialism: For all $(a, A), (b, B) \in \Omega^*$, $(a, A) \succeq (b, B) \Leftrightarrow [\sigma(A) > \sigma(B)$, or $\{(\sigma(A) = \sigma(B)$ and $(a, \{a\}) \succeq (b, \{b\})\}]$.

It should be clear that, without imposing any other condition on the extended preference ordering \succeq over Ω^* , the ordering \succeq^* over $\Omega^1 \times \mathcal{R}$ figured in Theorem 5.1 may not be representable by any numerical function. In order for \succeq^* to be representable by a numerical function, we consider the Continuity condition defined in the last section and obtain the following result.

Theorem 5.3. Suppose \succeq satisfies (DEN), (ARP), (COM), (SEP) and (CON). Then there exist a function $\sigma : K \rightarrow \mathcal{R}$, a continuous function $u : X \rightarrow \mathcal{R}$ and an ordering \succeq^* on $\mathcal{R} \times \mathcal{Z}$ such that

- (a) σ is countably additive; $\sigma(\emptyset) = 0$; For all $A, B \in K$, if $B \subset A$, then $\sigma(A) \geq \sigma(B)$;
For all $A, B \in K$, and all $x \in X$, $(x, A \cup \{x\}) \succeq (x, B \cup \{x\}) \Leftrightarrow \sigma(A) \geq \sigma(B)$.
- (b) For all $x, y \in X$, $u(x) \geq u(y) \Leftrightarrow (x, \{x\}) \succeq (y, \{y\})$;
- (c) For all $(x, A), (x, B) \in \Omega^*$, $(x, A) \succeq (y, B) \Leftrightarrow (u(x), \sigma(A)) \succeq^* (u(y), \sigma(B))$.
- (d) For all $(x, A), (y, B) \in \Omega^*$ and all $C \in K$, if $\sigma(A \cup C) = \sigma(A) + \sigma(C)$ and $\sigma(B \cup C) = \sigma(B) + \sigma(C)$, then $(u(x), \sigma(A)) \succeq^* (u(y), \sigma(B)) \Leftrightarrow (u(x), \sigma(A) + \sigma(C)) \succeq^* (u(y), \sigma(B) + \sigma(C))$.

Proof. Given Theorem 5.1, a similar method of proving Theorem 4.5 can be invoked to prove Theorem 5.3. ■

6 Concluding Remarks

In this paper, we have extended our previous framework for analyzing consequences and opportunities so as to accommodate the situations where consequential considerations and opportunity considerations actively interact. We have covered both the

cases of finite and infinite number of alternatives, and several representation results have been derived.

It is hoped that with this general framework, one can examine some classical problems like Arrow's impossibility theorem in social choice theory, pure theory of consumer's behavior, and competitive equilibrium mechanism in general equilibrium analysis, and derive some novel and useful insights. However, this avenue of research must be left for future exploration.

Endnotes

1. There is an extensive literature addressing non-welfaristic features of consequences. See, among others, Baharad and Nitzan [1], Bossert, Pattanaik and Xu [2], Dworkin [3], Gravel [4, 5], Pattanaik and Suzumura [6,7], Pattanaik and Xu [8, 9, 10], Sen [11, 12, 13, 14, 15, 16], Suzumura [17, 18], and Suzumura and Xu [19, 20].
2. According to (3.1), the function u can be construed as a utility function defined on the set of social states whereas the function f weighs the value of consequential states vis-à-vis the value of richness of opportunities which is measured in terms of the cardinality of opportunity sets. A similar remark applies to the function u in Theorems 4.5 and 5.3.

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