Expected utility without full transitivity*

Walter Bossert
Department of Economics and CIREQ
University of Montreal
P.O. Box 6128, Station Downtown
Montreal QC H3C 3J7
Canada
FAX: (+1 514) 343 7221
e-mail: walter.bossert@umontreal.ca

and

Kotaro Suzumura
School of Political Science and Economics
Waseda University
1-6-1 Nishi-Waseda
Shinjuku-ku, Tokyo 169-8050
Japan
FAX: (+81 3) 3311 5110
e-mail: ktr.suzumura@gmail.com

This version: May 26, 2014

Abstract. We generalize the classical expected-utility criterion by weakening transitivity to Suzumura consistency. In the absence of full transitivity, reflexivity and completeness no longer follow as a consequence of the system of axioms employed and a richer class of rankings of probability distributions results. This class is characterized by means of standard expected-utility axioms in addition to Suzumura consistency. An important feature of some members of our new class is that they allow us to soften the negative impact of well-known paradoxes without abandoning the expected-utility framework altogether. *Journal of Economic Literature* Classification No.: D81.

* Financial support from a Grant-in-Aid for Specially Promoted Research from the Ministry of Education, Culture, Sports, Science and Technology of Japan for the Project on Economic Analysis of Intergenerational Issues (grant number 22000001), the Fonds de Recherche sur la Société et la Culture of Québec, and the Social Sciences and Humanities Research Council of Canada is gratefully acknowledged.

1 Introduction

The formal treatment of the expected-utility criterion has a long-standing tradition in the theory of individual choice under uncertainty, going back as far as von Neumann and Morgenstern's (1944; 1947) seminal contribution. While numerous criticisms have been leveled at the descriptive suitability of expected-utility theory (often in the context of experimental studies), the criterion has proven to be rather robust in that it remains on a sound normative foundation. Nevertheless, inconsistencies with observable behavior such as the paradox formulated by Allais (1953) and analyzed in an experimental setting by Kahneman and Tversky (1979) constitute serious challenges that need to be responded to if (at least some form of) expected-utility theory is to continue to be an attractive option in descriptive contexts as well.

In an attempt to address paradoxes of this nature, several alternative theories have been developed over the years. These include Kahneman and Tversky's (1979) prospect theory and those proposed in the rapidly growing literature on behavioral approaches to economic decision-making; see Simon (1955), Camerer (1995) and Rabin (1998), for instance. Clearly, these references do not, by any means, constitute an exhaustive list but they serve as a starting point for an exploration of this evolving body of work.

The above-mentioned alternative approaches represent clear-cut departures from the expected-utility criterion. In contrast, the objective of this paper is to attempt to soften the negative impact of some established paradoxes without abandoning the expected-utility framework altogether. This is achieved by retaining most of the traditional expected-utility axioms but weakening transitivity to Suzumura consistency. As is well-known, transitivity (along with other standard expected-utility axioms) implies that the resulting criterion used to rank probability distributions is reflexive and complete—any two probability distributions can be ranked. In the absence of full transitivity, this implication is no longer valid and, thus, new decision criteria emerge as additional possibilities.

Many of the behavioral approaches alluded to above explicitly start out with the hypothesis that economic agents are not necessarily fully rational but that they make choices under what is often referred to as bounded rationality. An advantage of the theories developed in this context is their ability to explain specific observable patterns of behavior in a coherent manner. On the other hand, many of these models are restricted to rather specific situations and, thus, are difficult to justify as general methods to describe observed choices.

The notion of bounded rationality frequently appears in cases where the decision problem under consideration is deemed to be too complex to allow for full rationality in the traditional sense. This reasoning appears to apply analogously to situations in which the inherent complexity leads us to the assumption that economic agents may not be able to rank all possible probability distributions under consideration. Thus, this complexity argument can be invoked in support of our approach which allows for non-comparability as well. That completeness may be a rather strong assumption in the context of expected-utility theory has been argued in many earlier contributions—von Neumann and Morgenstern (1944; 1947) themselves make this point; other authors who question the completeness axiom include Thrall (1954), Luce and Raiffa (1957), Aumann (1962) and Dubra, Maccheroni and Ok (2004). After all, there are several instances where the imposition of completeness might create artificial puzzles and even impossibilities; the earlier contributions just cited are merely examples of such problems that may be triggered by the completeness assumption.

The class of decision rules that we characterize generalizes the expected-utility criterion in that some pairs of probability distributions may be considered to be non-comparable. Thus, this class is considerably richer than the traditional expected-utility criterion because it ranges from the classical (fully transitive) case itself to a minimal criterion that only imposes rankings for a small subset of the pairs of probability distributions. We emphasize that the novelty of our approach lies in the use of the weaker axiom of Suzumura consistency as opposed to transitivity; that reflexivity and completeness no longer follow is a consequence of this choice rather than an assumption. Although we do not impose full transitivity, we require that the ranking of probability distributions be Suzumura consistent, which can be seen as a rather plausible and appealing property; see also the discussion below.

An important consequence of making these more general (possibly incomplete) rules available is that paradoxes that involve rank reversals can be avoided by a suitable choice of a member of our class—namely, a generalized expected-utility criterion that is silent about the ranking of the types of pairs singled out in the paradox under consideration.

All but one of the axioms that we employ are standard in the literature on choice under uncertainty and in many other branches of the literature. In particular, we employ notions of solvability, monotonicity and independence. Loosely speaking, solvability is related to continuity properties, monotonicity rules out counter-intuitive rankings in relatively straightforward comparisons and independence is a separability condition.

The only axiom that is less familiar in the context of choice under uncertainty is Suzumura consistency and, for that reason, we discuss it in some detail. This coherence property of a binary relation was introduced by Suzumura (1976). It rules out all preference cycles that involve at least one instance of strict preference. Thus, Suzumura consistency is stronger than acyclicity which merely prohibits cycles such that all preferences involved are strict. Furthermore, Suzumura consistency is implied by transitivity. If a relation is reflexive and complete, Suzumura consistency also implies transitivity but this implication does not hold if reflexivity or completeness is violated. Quasi-transitivity, which demands that the asymmetric part of a relation be transitive, and Suzumura consistency are independent. Because Suzumura consistency is equivalent to transitivity in the presence of reflexivity and completeness, it can be considered a very natural weakening; note that quasi-transitivity fails to imply transitivity even if a relation is reflexive and complete and, of course, the same observation applies for acyclicity (which is weaker than quasi-transitivity).

Further forceful arguments in support of Suzumura consistency can be made. A well-known result due to Szpilrajn (1930) establishes that transitivity is sufficient for the existence of an ordering extension (that is, a reflexive, complete and transitive extension) of a relation. This is a fundamental result that has been applied in numerous settings. A remarkable strengthening of Szpilrajn's (1930) extension theorem is proven by Suzumura (1976) who shows that Suzumura consistency is necessary and sufficient for the existence of an ordering extension, thus providing a very clear demarcation line between the set of relations that can be extended to an ordering and those that cannot. This is another attractive feature of Suzumura consistency as compared to quasi-transitivity and acyclicity: neither of these properties can be used to obtain such an equivalence result. In addition, Bossert, Sprumont and Suzumura (2005) show that there exists a well-defined Suzumura-consistent closure of any relation, just as is the case for transitivity. No such closure operations exist in the cases of quasi-transitivity and acyclicity. These observations reinforce our statement that Suzumura consistency is indeed a natural weakening of transitivity, and a detailed analysis of the axiom is carried out in Bossert and Suzumura (2010) where we demonstrate the usefulness of this coherence property in numerous individual and collective decision problems including, among others, topics in revealed preference theory. See also Suzumura (1978) for the crucial service rendered by Suzumura consistency in the social-choice theoretic analysis of individual rights.

There are several studies that analyze expected-utility theory in the context of reflexive and transitive but not necessarily complete relations over probability distributions, such as those carried out by Aumann (1962) and by Dubra, Maccheroni and Ok (2004). However, these authors retain full transitivity as an assumption and, as a consequence, obtain results that are quite different in nature from ours.

Aumann (1962) considers rankings of probability distributions that are reflexive and transitive but not necessarily complete, in conjunction with a continuity property and a variant of the independence axiom. He shows that, under his assumptions, there exists an additive function such that if two distributions are indifferent, then they must generate the same expectation according to this function and, likewise, if a distribution is strictly preferred to another, the former is associated with a greater expectation than the latter. Clearly, the ranking generated by such a function is not necessarily complete and, as pointed out by Dubra, Maccheroni and Ok (2004, footnote 2), "... [this] approach falls short of yielding a representation theorem, for it does not *characterize* the preference relations under consideration."

Dubra, Maccheroni and Ok (2004) establish an expected multi-utility theorem that characterizes reflexive and transitive but possibly incomplete preferences on probability distributions by means of a continuity property and a version of the independence axiom. The idea underlying the multi-utility approach is that a reflexive and transitive dominance criterion can be established by means of a set of possible utility functions such that a distribution is considered at least as good as another if and only if the expectation according to the former is greater than or equal to that of the latter for all utility functions in this set.

Suzumura consistency is not employed in the above-described articles (or in any other contributions to expected-utility theory that we are aware of). Moreover, Aumann (1962) and Dubra, Maccheroni and Ok (2004) do not impose a property akin to the monotonicity condition alluded to earlier, which is a major reason why completeness does not follow from their axioms even in the presence of full transitivity. This is yet another feature that distinguishes these earlier approaches from ours.

In Section 2, we introduce our notation and basic definitions. Section 3 contains a preliminary characterization of decision criteria on the basis of our axioms without independence, followed by our main result. Section 4 provides some examples and concludes.

2 Definitions

Suppose there is a fixed finite set of alternatives $X = \{x_1, \ldots, x_n\}$, where $n \in \mathbb{N} \setminus \{1, 2\}$. We exclude the one-alternative and two-alternative cases because they are trivial. The set $\Delta = \{p \in \mathbb{R}^n_+ \mid \sum_{i=1}^n p_i = 1\}$ is the unit simplex in $\mathbb{R}^n_+$, interpreted as the set of all probability distributions on X. For all $i \in \{1, \ldots, n\}$, the i^{th} unit vector in $\mathbb{R}^n$ is denoted by e^i.

A (binary) relation on Δ is a set $\succsim \subseteq \Delta^2$. As usual, the symmetric and asymmetric parts $\sim$ and $\succ$ of $\succsim$ are defined by letting, for all $p, q \in \Delta$,

$$p \sim q \;\Leftrightarrow\; p \succsim q \text{ and } q \succsim p$$

and

$$p \succ q \;\Leftrightarrow\; p \succsim q \text{ and } \neg(q \succsim p).$$

We interpret the relation $\succsim$ as the preference relation (or the decision rule) used by an agent to rank probability distributions. The relations $\sim$ and $\succ$ are the corresponding indifference relation and strict preference relation.

The transitive closure $\overline{\succsim}$ of $\succsim$ is defined by letting, for all $p, q \in \Delta$,

$$p \,\overline{\succsim}\, q \;\Leftrightarrow\; \begin{array}{l}\text{there exist } K \in \mathbb{N} \text{ and } r^0, \ldots, r^K \in \Delta \text{ such that}\\ p = r^0 \text{ and } r^{k-1} \succsim r^k \text{ for all } k \in \{1, \ldots, K\} \text{ and } r^K = q.\end{array}$$

We now introduce the properties of the decision criterion $\succsim$ that are of importance in this paper. The first three of these are simply the properties that define an ordering. Note that reflexivity and completeness are usually not imposed when formulating one of the versions of the classical expected-utility theorem; they are implied by the set of axioms employed in the requisite result when transitivity is one of these axioms. We do not require reflexivity and completeness in our generalized expected-utility theorem either. Because the full force of transitivity is not imposed, a more general class of (not necessarily reflexive and complete) decision criteria can be obtained.

Reflexivity. For all $p \in \Delta$, $p \succsim p$.

Completeness. For all $p, q \in \Delta$ such that $p \neq q$, $p \succsim q$ or $q \succsim p$.

Transitivity. For all $p, q, r \in \Delta$, $[p \succsim q \text{ and } q \succsim r] \;\Rightarrow\; p \succsim r$.

The next three properties are standard in decision theory as well as, suitably reformulated, in numerous other areas within microeconomic theory. The first of these amounts to a continuity condition, the second ensures that the direction of preference is in accord with the interpretation of the relation $\succsim$ as a decision criterion for choice under uncertainty, and the third is a separability property. The special role played by the alternatives labeled x_1 and x_n in the axioms of solvability and monotonicity does not involve any loss of generality. All that matters is that there are two certain alternatives that are strictly ranked and we simply assume that these alternatives are given by x_1 and x_n.

Solvability. For all $p \in \Delta$, there exists $\alpha \in [0,1]$ such that $p \sim (\alpha, 0, \ldots, 0, 1-\alpha)$.

Monotonicity. For all $\alpha, \beta \in [0,1]$,

$$(\alpha, 0, \ldots, 0, 1-\alpha) \succsim (\beta, 0, \ldots, 0, 1-\beta) \;\Leftrightarrow\; \alpha \geq \beta.$$

Independence. For all $p, q \in \Delta$ and for all $\alpha, \beta, \gamma \in [0,1]$, if $p \sim (\alpha, 0, \ldots, 0, 1-\alpha)$ and $q \sim (\beta, 0, \ldots, 0, 1-\beta)$, then

$$\gamma p + (1-\gamma) q \sim \gamma(\alpha, 0, \ldots, 0, 1-\alpha) + (1-\gamma)(\beta, 0, \ldots, 0, 1-\beta).$$

Our formulation of the independence axiom differs from some of the traditional variants in some respects. One of the standard versions requires that an indifference between two distributions p and q implies that any convex combination of p and any distribution r with weights γ and $1-\gamma$ is indifferent to the convex combination of q and r with the same weights γ and $1-\gamma$. In the presence of transitivity (or merely transitivity of the indifference relation $\sim$), our axiom is implied by this alternative property because it restricts the requisite implication to a smaller set of pairs of distributions. The reason why we employ this alternative formulation is that we want to be able to arrive at a characterization result without having to impose transitivity of $\sim$. As is well-known (see, for instance, Luce's, 1956, much-cited coffee-sugar example provides a powerful argument against the use of transitive indifference: a decision maker may find it very difficult to perceive small differences and, thus, the indifference relation may very well fail to be transitive. If one is willing to require $\sim$ to be transitive, a more restrictive class of decision rules is characterized in the presence of our remaining axioms; in this case, all indifferences according to the expected-utility criterion have to be respected. Details on this alternative result are available from the authors on request.

The remaining axiom is Suzumura consistency. Recall that, in the presence of reflexivity and completeness, Suzumura consistency and transitivity are equivalent but, because our new result does not involve reflexivity and completeness, transitivity is not implied. Again, see Suzumura (1976) and Bossert and Suzumura (2010) for more detailed discussions.

Suzumura consistency. For all $p, q \in \Delta$, $p \,\overline{\succsim}\, q \;\Rightarrow\; \neg(q \succ p)$.

Unlike Kreps (1988) and other authors, we treat $\succsim$ as the primitive concept rather than $\succ$. As is standard when $\succ$ is considered to be the primary relation, Kreps (1988) imposes asymmetry and negative transitivity on $\succ$. If the relation $\succsim$ is required to be an ordering (that is, reflexive, complete and transitive), it does not matter whether we start out with $\succsim$ or with the associated asymmetric part $\succ$ as the primitive notion because the conjunction of asymmetry and negative transitivity of $\succ$ implies that the corresponding relation $\succsim$ is an ordering; see Kreps (1988, pp.9–10). Thus, adopting Kreps's (1988) setting would result in an immediate conflict with our main objective of examining the consequences of weakening the transitivity requirement in the context of expected-utility theory.

3 Suzumura-consistent expected utility

The main result of this paper establishes that if transitivity is weakened to Suzumura consistency, a class of generalized expected-utility criteria is characterized. These relations allow for violations of reflexivity or completeness in some situations. The preferences imposed by solvability and monotonicity continue to be required but, because full transitivity is no longer used, any additional pairs that belong to the expected-utility relation may or may not be included. Thus, the new class contains as special cases the standard (reflexive and complete) expected-utility criterion as the "maximal" relation and the one where the only preferences are those imposed by solvability and monotonicity as the "minimal" relation satisfying the axioms. In particular, this means that any other pair that is weakly (strictly) ranked by the expected-utility criterion may either be weakly (strictly) ranked or non-comparable according to our generalization. Thus, undesirable observations such as the Allais paradox (Allais, 1953) can be ameliorated by replacing at least one of the two counter-intuitive preferences with non-comparability. We illustrate this feature in our concluding section.

As a first step, we characterize the class of all decision criteria that satisfy Suzumura consistency, solvability and monotonicity. This theorem is of some interest in its own right: even in the absence of independence (the quintessential condition that underlies the expected-utility criterion), weakening transitivity to Suzumura consistency yields a precisely defined class of decision rules. We also use parts of its proof in establishing our main result.

Theorem 1 *A relation* $\succsim$ *on* Δ *satisfies Suzumura consistency, solvability and monotonicity if and only if there exists a function* $\varphi\colon \Delta \to [0,1]$ *such that the pair* $(\succsim, \varphi)$ *satisfies*

$$\textit{(i)} \quad [(\alpha, 0, \ldots, 0, 1-\alpha) \succsim (\beta, 0, \ldots, 0, 1-\beta) \;\Leftrightarrow\; \alpha \geq \beta] \quad \textit{for all } \alpha, \beta \in [0,1];$$

$$\textit{(ii)} \quad p \sim (\varphi(p), 0, \ldots, 0, 1-\varphi(p)) \quad \textit{for all } p \in \Delta;$$

$$\textit{(iii)} \quad p \sim q \;\Rightarrow\; \varphi(p) = \varphi(q) \quad \textit{for all } p, q \in \Delta;$$

$$\textit{(iv)} \quad p \succ q \;\Rightarrow\; \varphi(p) > \varphi(q) \quad \textit{for all } p, q \in \Delta.$$

Proof. 'If.' Solvability and monotonicity follow immediately from (i) and (ii). Suppose, by way of contradiction, that Suzumura consistency is violated. Then there exist $p, q \in \Delta$ such that $p \,\overline{\succsim}\, q$ and $q \succ p$. Thus, by definition of the transitive closure, there exist $K \in \mathbb{N}$ and $r^0, \ldots, r^K \in \Delta$ such that $p = r^0$ and $r^{k-1} \succsim r^k$ for all $k \in \{1, \ldots, K\}$ and $r^K = q$. Consider any $k \in \{1, \ldots, K\}$. If $r^{k-1} \sim r^k$, it follows that

$$\varphi(r^{k-1}) = \varphi(r^k)$$

because of (iii). If $r^{k-1} \succ r^k$, (iv) implies

$$\varphi(r^{k-1}) > \varphi(r^k).$$

Thus, for all $k \in \{1, \ldots, K\}$, we have

$$\varphi(r^{k-1}) \geq \varphi(r^k).$$

Combining these inequalities for all $k \in \{1, \ldots, K\}$ and using $p = r^0$ and $r^K = q$, it follows that

$$\varphi(p) \geq \varphi(q). \tag{1}$$

By (iv), $q \succ p$ implies

$$\varphi(q) > \varphi(p),$$

contradicting (1). Thus, Suzumura consistency is satisfied.

'Only if.' Part (i) follows immediately from monotonicity.

To prove (ii), let $p \in \Delta$ and $\alpha \in [0, 1]$ be such that

$$p \sim (\alpha, 0, \ldots, 0, 1 - \alpha). \tag{2}$$

The existence of α is guaranteed by solvability. Furthermore, α is unique. To see this, suppose, by way of contradiction, that there exists $\beta \in [0, 1] \setminus \{\alpha\}$ such that

$$p \sim (\beta, 0, \ldots, 0, 1 - \beta). \tag{3}$$

If $\beta > \alpha$, we have

$$(\beta, 0, \ldots, 0, 1 - \beta) \succ (\alpha, 0, \ldots, 0, 1 - \alpha) \tag{4}$$

by monotonicity and, by (2) and (3),

$$(\alpha, 0, \ldots, 0, 1 - \alpha) \succsim (\beta, 0, \ldots, 0, 1 - \beta)$$

which, together, with (4), leads to a contradiction to Suzumura consistency. The same argument applies if $\beta < \alpha$. Thus, α must be unique for p and we can write it as a function $\varphi: \Delta \to [0, 1]$, that is,

$$p \sim (\varphi(p), 0, \ldots, 0, 1 - \varphi(p)) \quad \text{for all } p \in \Delta. \tag{5}$$

This establishes the existence of a function φ such that (ii) is satisfied.

Now we prove (iii). Suppose, by way of contradiction, that there exist $p, q \in \Delta$ such that $p \sim q$ and $\varphi(p) \neq \varphi(q)$. Without loss of generality, suppose that $\varphi(q) > \varphi(p)$. By monotonicity,

$$(\varphi(q), 0, \ldots, 0, 1 - \varphi(q)) \succ (\varphi(p), 0, \ldots, 0, 1 - \varphi(p)). \tag{6}$$

Furthermore, because

$$(\varphi(p), 0, \ldots, 0, 1 - \varphi(p)) \sim p$$

and

$$p \sim q$$

and

$$q \sim (\varphi(q), 0, \ldots, 0, 1 - \varphi(q)),$$

it follows that

$$(\varphi(p), 0, \ldots, 0, 1-\varphi(p)) \,\overline{\succsim}\, (\varphi(q), 0, \ldots, 0, 1-\varphi(q))$$

which, together with (6), contradicts Suzumura consistency.

To prove (iv), suppose, by way of contradiction, that there exist $p, q \in \Delta$ such that $p \succ q$ and $\varphi(p) \leq \varphi(q)$. By monotonicity,

$$(\varphi(q), 0, \ldots, 0, 1-\varphi(q)) \succsim (\varphi(p), 0, \ldots, 0, 1-\varphi(p)).$$

Furthermore, because we also have

$$q \sim (\varphi(q), 0, \ldots, 0, 1-\varphi(q))$$

and

$$(\varphi(p), 0, \ldots, 0, 1-\varphi(p)) \sim p,$$

it follows that $q \,\overline{\succsim}\, p$ which, together with $p \succ q$, leads to a contradiction of Suzumura consistency. ■

Now we can state and prove our main result.

Theorem 2 *A relation* $\succsim$ *on* Δ *satisfies Suzumura consistency, solvability, monotonicity and independence if and only if there exists a function* $U\colon X \to [0,1]$ *such that the pair* $(\succsim, U)$ *satisfies*

$$\textit{(0)} \quad U(x_1) = 1 \;\; \textit{and} \;\; U(x_n) = 0;$$

$$\textit{(i)} \quad [(\alpha, 0, \ldots, 0, 1-\alpha) \succsim (\beta, 0, \ldots, 0, 1-\beta) \;\Leftrightarrow\; \alpha \geq \beta] \quad \textit{for all } \alpha, \beta \in [0,1];$$

$$\textit{(ii)} \quad p \sim \left(\sum_{i=1}^{n} p_i U(x_i), 0, \ldots, 0, 1 - \sum_{i=1}^{n} p_i U(x_i)\right) \quad \textit{for all } p \in \Delta;$$

$$\textit{(iii)} \quad p \sim q \;\Rightarrow\; \sum_{i=1}^{n} p_i U(x_i) = \sum_{i=1}^{n} q_i U(x_i) \quad \textit{for all } p, q \in \Delta;$$

$$\textit{(iv)} \quad p \succ q \;\Rightarrow\; \sum_{i=1}^{n} p_i U(x_i) > \sum_{i=1}^{n} q_i U(x_i) \quad \textit{for all } p, q \in \Delta.$$

Proof. 'If.' Solvability and monotonicity follow from (i) and (ii). That Suzumura consistency is satisfied is an immediate consequence of substituting $\varphi(p) = \sum_{i=1}^{n} p_i U(x_i)$ for all $p \in \Delta$ and applying the requisite result of the 'if' part of Theorem 1.

To prove that the members of the class of decision rules identified in the statement of Theorem 2 satisfy independence, suppose $p, q \in \Delta$ and $\alpha, \beta \in [0,1]$ are such that $p \sim (\alpha, 0, \ldots, 0, 1-\alpha)$ and $q \sim (\beta, 0, \ldots, 0, 1-\beta)$. By (i) and (ii) and the uniqueness of α and β, this means that $\alpha = \sum_{i=1}^{n} p_i U(x_i)$ and $\beta = \sum_{i=1}^{n} q_i U(x_i)$. We have to show that, for all $\gamma \in [0,1]$,

$$\gamma p + (1-\gamma) q \;\sim\; \gamma \left(\sum_{i=1}^{n} p_i U(x_i), 0, \ldots, 0, 1 - \sum_{i=1}^{n} p_i U(x_i)\right) + (1-\gamma) \left(\sum_{i=1}^{n} q_i U(x_i), 0, \ldots, 0, 1 - \sum_{i=1}^{n} q_i U(x_i))\right)$$

or, equivalently,

$$\gamma p+(1-\gamma)q \sim \left(\sum_{i=1}^{n}(\gamma p_i+(1-\gamma)q_i)U(x_i),0,\ldots,0,1-\sum_{i=1}^{n}(\gamma p_i+(1-\gamma)q_i)U(x_i)\right). \quad (7)$$

Letting $\gamma p+(1-\gamma)q$ play the role of p in (ii), (7) follows.

'Only if.' The function φ can be constructed as in the proof of Theorem 1. Now define $U\colon X \to [0,1]$ by letting

$$U(x_i)=\varphi(e^i) \quad \text{for all } i \in \{1,\ldots,n\}. \tag{8}$$

Substituting $p=e^1$ in (5) and using (8), it follows that

$$e^1=(1,0,\ldots,0) \sim (U(x_1),0,\ldots,0,1-U(x_1))$$

and $U(x_1)=1$ follows immediately from uniqueness which, in turn, follows from Suzumura consistency and monotonicity. That $U(x_n)=0$ follows analogously and, thus, the proof of (0) is complete.

Part (i) is an immediate consequence of monotonicity.

In view of parts (ii), (iii) and (iv) of Theorem 1, the proof is complete once we show that

$$\varphi(p)=\sum_{i=1}^{n}p_iU(x_i) \quad \text{for all } p \in \Delta. \tag{9}$$

Let $p,q \in \Delta$ and $\gamma \in [0,1]$. By definition of φ, we have

$$p \sim (\varphi(p),0,\ldots,0,1-\varphi(p)) \;\text{ and }\; q \sim (\varphi(q),0,\ldots,0,1-\varphi(q)).$$

By independence,

$$\begin{aligned}\gamma p+(1-\gamma)q &\sim \gamma(\varphi(p),0,\ldots,0,1-\varphi(p))+(1-\gamma)(\varphi(q),0,\ldots,0,1-\varphi(q))\\ &= (\gamma\varphi(p)+(1-\gamma)\varphi(q),0,\ldots,0,\gamma(1-\varphi(p))+(1-\gamma)(1-\varphi(q)))\\ &= (\gamma\varphi(p)+(1-\gamma)\varphi(q),0,\ldots,0,1-[\gamma\varphi(p)+(1-\gamma)\varphi(q)]).\end{aligned}$$

By definition of φ, this means that

$$\varphi(\gamma p+(1-\gamma)q)=\gamma\varphi(p)+(1-\gamma)\varphi(q). \tag{10}$$

We now use the definition of U in (8) and the functional equation (10) to prove (9). The proof proceeds by induction on the number of positive components of p, that is, on the cardinality of the set $\{i \in \{1,\ldots,n\} \mid p_i>0\}$. This step in our proof is borrowed from Kreps's (1988) proof of a version of the classical expected-utility theorem.

If $\{i \in \{1,\ldots,n\} \mid p_i>0\}$ contains a single element j, it follows that $p=e^j$. Clearly, $\varphi(e^j)=\sum_{i=1}^{n}p_iU(x_i)=U(x_j)$ in this case and (9) is satisfied.

Now let $1 < m \leq n$ and suppose (9) is satisfied for all probability distributions in Δ with $m-1$ positive components. Let p be such that p has m positive components and let $i \in \{1, \ldots, n\}$ be such that $p_i > 0$. Define a distribution $q \in \Delta$ by

$$q_j = \begin{cases} 0 & \text{if } j = i, \\ \frac{p_j}{1-p_i} & \text{if } j \neq i. \end{cases}$$

By definition, q has $m-1$ positive components and we can express p as

$$p = p_i e^i + (1-p_i)q.$$

By (10),

$$\varphi(p) = p_i \varphi(e^i) + (1-p_i)\varphi(q)$$

and, using (8) and applying the induction hypothesis to q, it follows that

$$\varphi(p) = p_i U(x_i) + (1-p_i) \sum_{\substack{j=1 \\ j \neq i}}^{n} \frac{p_j}{1-p_i} U(x_i) = \sum_{i=1}^{n} p_i U(x_i),$$

as was to be shown. ■

In most of the formulations of the classical expected-utility theorem, reflexivity and completeness are implied due to the presence of transitivity in the set of axioms employed. Clearly, this is not the case if transitivity is weakened to Suzumura consistency. Moreover, not even the restriction of $\succsim$ to pairs of unit vectors needs to be an ordering. That is, Suzumura consistency in conjunction with the remaining axioms of Theorem 2 is not sufficient to guarantee that certain alternatives (the elements of X) are fully ordered.

4 Examples and concluding remarks

As mentioned earlier, the classical expected-utility criterion is a special case of the class characterized in Theorem 2. Another special case is obtained if no preferences are added to those imposed by (i) and (ii), that is, the case in which $\succsim$ is defined by letting

$$[(\alpha, 0, \ldots, 0, 1-\alpha) \succsim (\beta, 0, \ldots, 0, 1-\beta) \Leftrightarrow \alpha \geq \beta] \quad \text{for all } \alpha, \beta \in [0,1]$$

and

$$p \sim \left(\sum_{i=1}^{n} p_i U(x_i), 0, \ldots, 0, 1 - \sum_{i=1}^{n} p_i U(x_i) \right) \quad \text{for all } p \in \Delta.$$

To see how some members of the class characterized in Theorem 2 allow us to ameliorate the supposedly disagreeable consequences of observations such as the Allais paradox, consider the following example. Suppose we have a set of alternatives $X = \{5, 1, 0\}$, where $x_1 = 5$ stands for receiving five million dollars, $x_2 = 1$ means receiving one million dollars and $x_3 = 0$ is an alternative in which the agent receives zero. Experimental evidence (see Kahneman and Tversky, 1979, for instance) suggests that numerous subjects express something resembling the following rankings of specific probability distributions. Consider the distributions $p = (0, 1, 0)$ and $q = (0.1, 0.89, 0.01)$ on the one hand, and the pair of distributions $p' = (0, 0.11, 0.89)$ and $q' = (0.1, 0, 0.9)$ on the other. Many subjects appear to rank p as being better than q and q' as being better than p'. This is inconsistent with classical expected utility theory because, for any function $U: X \to [0, 1]$ such that $U(5) = 1$ and $U(0) = 0$, $p \succ q$ entails $U(1) > 0.1 + 0.89\, U(1)$ and, thus, $U(1) > 1/11$, whereas $q' \succ p'$ implies $0.1 > 0.11\, U(1)$ and, thus, $U(1) < 1/11$. If, however, a generalized expected-utility criterion that allows for incompleteness is employed and the subjects are given the option of treating two distributions as non-comparable, it may very well be the case that, according to the decisions of some experimental subjects, p is ranked higher than q and p' and q' are non-comparable (or p and q are non-comparable and q' is preferred to p', or both pairs are non-comparable). Of course, the so-called paradox persists if the original rankings are retained even in the presence of a non-comparability option. But it seems to us that the use of an incomplete generalized expected-utility criterion may considerably reduce the instances of dramatically conflicting pairwise rankings without abandoning the core principles of expected-utility theory altogether.

References

Allais, M. (1953), Le comportement de l'homme rationnel devant le risque: critique des postulats et axiomes de l'école Americaine, *Econometrica* **21**, 503–546.

Aumann, R.J. (1962), Utility theory without the completeness axiom, *Econometrica* **30**, 445–462.

Bossert, W., Y. Sprumont and K. Suzumura (2005), Consistent rationalizability, *Economica* **72**, 185–200.

Bossert, W. and K. Suzumura (2010), *Consistency, Choice, and Rationality,* Harvard University Press, Cambridge, MA.

Camerer, C. (1995), Individual decision making, in: J.H. Kagel and A.E. Roth (eds.), *Handbook of Experimental Economics,* Princeton University Press, Princeton, NJ, 587–704.

Dubra, J., F. Maccheroni and E.A. Ok (2004), Expected utility theory without the completeness axiom, *Journal of Economic Theory* **115**, 118–133.

Kahneman, D. and A. Tversky (1979), Prospect theory: an analysis of decision under risk, *Econometrica* **47**, 263–291.

Kreps, D.M. (1988), *Notes on the Theory of Choice,* Westview Press, Boulder, CO.

Luce, R.D. (1956), Semiorders and the theory of utility discrimination, *Econometrica* **24**, 178–191.

Luce, R.D. and H. Raiffa (1957), *Games and Decisions,* Wiley, New York.

Rabin, M. (1998), Psychology and economics, *Journal of Economic Literature* **36**, 11–46.

Simon, H.A. (1955), A behavioral model of rational choice, *Quarterly Journal of Economics* **69**, 99–118.

Suzumura, K. (1976), Remarks on the theory of collective choice, *Economica* **43**, 381–390.

Suzumura, K. (1978), On the consistency of libertarian claims, *Review of Economic Studies* **45**, 329–342.

Szpilrajn, E. (1930), Sur l'extension de l'ordre partiel, *Fundamenta Mathematicae* **16**, 386–389.

Thrall, R.M. (1954), Applications of multidimensional utility theory, in: R.M. Thrall, C.H. Coombs and R.L. Davis (eds.), *Decision Processes,* Wiley, New York, 181–186.

von Neumann, J. and O. Morgenstern (1944; second ed. 1947), *Theory of Games and Economic Behavior,* Princeton University Press, Princeton, NJ.