Choice, Opportunities, and Procedures: Collected Papers of Kotaro Suzumura

Part II Equity, Efficiency, and Sustainability

Kotaro Suzumura
Institute of Economic Research, Hitotsubashi University
Kunitatchi, Tokyo 186-8603, Japan
suzumura@ier.hit-u.ac.jp

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Chapter 5
On Pareto-Efficiency and the No-Envy Concept of Equity*

1 Introduction

In dividing some fixed amount of resources among a fixed number of individuals, the exclusive reliance on the Pareto-efficiency criterion will be of little help, since *too many* feasible divisions will be Pareto-efficient under the standard environmental conditions. The introduction of an additional criterion of equity to the effect that a division is *equitable* if and only if no individual envies the position of another individual when the specified division is implemented will substantially narrow down the range of eligible divisions.[1] But the joint use of the Pareto-efficiency and the no-envy concept of equity encounters difficulty when the amount to be divided depends upon the contribution made by individuals, among whom ability differential prevails, since there may then be *no* eligible division even under the standard environmental conditions.[2] One possible response to this dilemma would be to observe that, from the viewpoint of moral philosophy, it is not altogether clear whether a concept of equity based on envy can be ethically relevant in the first place and to wash one's hands of the business. The second and arguably more "fruitful" response would be to propose a modified definition of equity which, coupled with the Pareto-efficiency, provides us with an alternative definition of eligibility. In the literature, we have abundantly many proposals to this effect: *wealth-fairness* (Varian [20; 21]), *income-fairness* (Varian [20] and Pazner [9]), *balanced-with-respect-to-envy-justice* (Daniel [5]), *egalitarian-equivalence* (Pazner and Schmeidler [11]) and *fairness-equivalence*

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[1] This concept of equity is due originally to Foley [7, Sect. IV]. See also Pazner and Schmeidler [10] and Varian [20].

[2] This dilemma was first exposed by Pazner and Schmeidler [10].

(Pazner [9]).[3] The third response would be to imbed the proposed concept of fairness as equity-cum-efficiency into the conceptual framework of social choice theory with a view of evaluating how the identified difficulty would be located in the perennial enigma of designing "satisfactory" collective choice mechanisms. The present paper is an attempt in this third category of exercises which are related to equity, envy and efficiency. We work with a model of social choice along the line of Arrow [3, Chap. VIII, Sect. IV. 4], Sen [16, Chap. 9*] and Suppes [18], in which the aggregation problem is to map each profile of individuals' extended orderings of $X \times N$ into a choice function on the family of subsets of X, where X and N denote the set of social states and the set of individuals, respectively. In this set-up, the viability of Foley's [7] original definition of fairness will be critically examined.

2 Fairness-as-No-Envy-cum-Efficiency and a Generalized Collective Choice Rule

2.1. As Varian [21, p.240] has aptly observed, "[t]he theory of fairness ... is founded in the notion of 'extended sympathy' and in the idea of 'symmetry' in the treatment of agents. ... In effect, we are asking each agent to put himself in the position of each other agents to determine if that is a better or a worse position than the one he is now in." To formalize this foundation of the theory of fairness, let X and $N = \{1, 2, \dots, n\}$ $(2 \leq n < +\infty)$ stand, respectively, for the set of all conceivable social states and the set of individuals, a social state being the complete description of the relevant aspects of the world. For each $i \in N$, we describe his/her views on the society by an extended preference ordering $\widetilde{R}_i$ on the Cartesian product $X \times N$,

$$((x, j), (y, k)) \in \widetilde{R}_i$$

denoting the fact that being in the position of individual j in the social state x is at least as good as being in the position of individual k in the social state y according to i's view.[4] A list of extended preference orderings, one ordering for each individual, will be called a *profile* and alternative profiles will be indexed by $\alpha, \beta, \dots$ like $\alpha = (\widetilde{R}_1^\alpha, \widetilde{R}_2^\alpha, \dots, \widetilde{R}_n^\alpha)$, $\beta = (\widetilde{R}_1^\beta, \widetilde{R}_2^\beta, \dots, \widetilde{R}_n^\beta)$ and so on. The set of all logically possible profiles will be denoted by $\mathscr{A}$, while the set of all non-empty finite subsets of X will be written as $\mathscr{S}$, each and every $S \in \mathscr{S}$ being construed to represent a set of available states under the specified environmental conditions.

2.2. Take a profile $\alpha = (\widetilde{R}_1^\alpha, \widetilde{R}_2^\alpha. \dots, \widetilde{R}_n^\alpha) \in \mathscr{A}$ and $S \in \mathscr{S}$ and fix them for the time being. For each individual $i \in N$, let i's *subjective preference ordering* R_i^α be defined by

$$R_i^\alpha = \{(x, y) \in X \times X \mid ((x, i), (y, i)) \in \widetilde{R}_i^\alpha\}, \tag{1}$$

[3]These proposals were succinctly surveyed and critically evaluated by Pazner [9] and Sen [17, Sect. 5].

[4]Being an ordering, $\widetilde{R}_i$ is connected, reflexive and transitive binary relation on $X \times N$.

in terms of which the *α-Paretian quasi-ordering* R^α_N will be defined by

$$R^\alpha_N = \cap_{i\in N} R^\alpha_i. \tag{2}$$

The set of all α-Pareto-efficient states in S will then be denoted as follows:

$$E^\alpha_f(S) = \{x \in S \mid\sim [\exists y \in S : (y, x) \in P(R^\alpha_N)]\}, \tag{3}$$

where and hereafter $P(\cdot)$ will denote an operator giving the asymmetric part of the binary relation in the parenthesis and $\sim$ denotes logical negation.

Following Foley's [7] classical definition, we say that individual $i \in N$ envies individual $j \in N$ at $x \in X$ when the profile α prevails if and only if $((x, j), (x, i)) \in P(\widetilde{R}^\alpha_i)$ holds true. We say that x is α-equitable if and only if nobody envies other individuals at x when the profile α prevails. The set of all α-equitable states in $S \in \mathscr{S}$ may be written as follows:

$$E^\alpha_q(S) = \{x \in S \mid \forall i, j \in N : ((x, i), (x, j)) \in \widetilde{R}^\alpha_i\}. \tag{4}$$

If a state in S is simultaneously α-Pareto-efficient and α-equitable, it is said to be *α-fair* in S. The set of all α-fair states in S, to be called the α-fair set in S, will be denoted by

$$F^\alpha(S) = E^\alpha_f(S) \cap E^\alpha_q(S). \tag{5}$$

2.3. The first point to be clarified about the α-fair set in S is that it may well be empty. Indeed, it is even possible that $E^\alpha_q(X) = \varnothing$ for some $\alpha \in \mathscr{A}$, namely, there may exist no α-equitable state wheresoever. Even if α-Pareto-efficiency and α-equity are *individually* self-consistent (in the sense that they may respectively be satisfied), they may well be *jointly* incompatible as is easily exemplified as follows.

EXAMPLE 1. Let $X = \{x, y\}$ and $N = \{1, 2\}$. Let a profile α be specified by

$$\widetilde{R}^\alpha_1 : (y, 2), (y, 1), (x, 1), (x, 2),$$
$$\widetilde{R}^\alpha_2 : (y, 2), (y, 1), (x, 2), (x, 1).$$ [5]

Clearly $E^\alpha_q(\{x, y\}) = \{x\}$ and $P(R^\alpha_N) = \{(y, x)\}$, so that we have $F^\alpha(\{x, y\}) = \varnothing$. ■

Simple though this example is, it may serve us well to expose several important features of the α-fairness concept. As an auxiliary step, let R^α_{ij} be defined by

$$R^\alpha_{ij} = \{(x, y) \in X \times X \mid ((x, j), (y, j)) \in \widetilde{R}^\alpha_i\}. \tag{6}$$

Clearly it is true that $R^\alpha_i = R^\alpha_{ii}$ for all $\alpha \in \mathscr{A}$ and all $i \in N$. Notice that $(x, y) \in R^\alpha_i$ means that individual i thinks that x is no worse for him than y, while $(x, y) \in R^\alpha_{ij}$ means that i thinks that it is no worse for j to be in x rather than in y. With this interpretation of R^α_i and R^α_{ij}, it seems fairly natural in the context where we talk about

[5] Preference orderings will be written horizontally with more preferred state-individual combination to the left of less preferred, indifferent combinations (if any) being put together by square brackets.

the welfare judgements based on the extended sympathy that we require the fulfillment of the following axiom, which is due to Sen [16, p. 156].[6]

AXIOM OF IDENTITY.

$$\forall i, j \in N : R_{ij}^{\alpha} = R_j^{\alpha}.$$

It is well recognized that the lack of the sympathetic acceptance of other's subjective preferences, namely, the invalidity of the axiom of identity, causes many logical difficulty in the exercise of aggregating profiles of extended preference orderings.[7] Notice, however, that the profile specified in the Example 1 does satisfy the axiom of identity, so that the problem identified by this example emerges even if the sympathetic identification prevails among individuals.

Our second remark on the α-fairness concept concerns with the contrast between the justice concept thereby implied and the traditional rival justice concepts, i.e., the *Rawlsian leximin justice* and the *Benthamite utilitarian justice*. Notice that in the situation specified by the Example 1 *both* a Rawlsian and a Benthamite would assert that y is more just than x — assuming for the sake of gaining comparability that 1 and 2 have interpersonally fully comparable cardinal representation of $\widetilde{R}_1^{\alpha}$ and $\widetilde{R}_2^{\alpha}$, respectively, while $E_q^{\alpha}(\{x, y\}) = \{x\}$ would force one to say that x is more equitable than y. The ethical appeal of the fairness-as-no-envy approach seems to be rather fragile indeed.

3 On the Possibility of Foley-Fair Collective Choice Rules

3.1. Our problem is to design a "fair" generalized collective choice rule Ψ, GCCR for short, which amalgamates each profile $\alpha \in \mathscr{A}$ of extended preference orderings into a social choice function $C^{\alpha} = \Psi(\alpha)$ on $\mathscr{S}$ such that, for each set $S \in \mathscr{S}$ of available social states, $C^{\alpha}(S)$ denotes the non-empty set of chosen states reflecting a "fair" amalgamation of α we have started from. In view of the possible non-existence of α-fair states for some $\alpha \in \mathscr{A}$, care should be taken with the sense in which we mean a GCCR to be "fair". One sense which naturally suggests itself is to require that Ψ satisfies the following condition.

FAIRNESS EXTENSION (FE). *For each admissible profile α, $C^{\alpha} = \Psi(\alpha)$ satisfies $F^{\alpha}(S) = C^{\alpha}(S)$ whenever $S \in \mathscr{S}$ is such that $F^{\alpha}(S) \neq \varnothing$.*

3.2. Collective choice is a repeated exercise in changing environments and one naturally feels that successive choices made should satisfy some "reasonable" choice-consistency condition. A choice-consistency condition which is deeply rooted in the

[6]What the axiom of identity requires is that "placing oneself in the position of the other should involve not merely having the latter's objective circumstances but also identifying oneself with the other in terms of his subjective features" (Sen [16, pp. 149-150]). It is debatable, however, if indeed we need literal transformation of subjective features so as to comply with the requirement of the axiom. On this and related points, the interested readers are referred to Suzumura [19].

[7]See, for example, Sen [16, pp. 149-150] and Suzumura [19].

Arrovian social choice theory is that of collective full-rationality (FR), which requires that we may construe a choice function C to describe a behaviour of optimizing fully consistent collective preference relation. Formally C satisfies the condition FR if and only if there exists a preference ordering R_C on X satisfying

$$\forall S \in \mathscr{S} : C(S) = \{x \in S \mid \forall y \in S : (x, y) \in R_C\}.$$

It was shown by Arrow [2] that a choice function C on $\mathscr{S}$ satisfies this condition FR if and only if C satisfies the following axiom.

ARROW'S AXIOM (AA).

$$\forall S_1, S_2 \in \mathscr{S} : S_1 \subset S_2 \Rightarrow [S_1 \cap C(S_2) = \varnothing \vee S_1 \cap C(S_2) = C(S_1)].$$

The following two axioms, which were found useful in various social and individual choice contexts, provide a natural decomposition of Arrow's axiom.

CHERNOFF'S AXIOM (CA).

$$\forall S_1, S_2 \in \mathscr{S} : S_1 \subset S_2 \Rightarrow [S_1 \cap C(S_2) = \varnothing \vee S_1 \cap C(S_2) \subset C(S_1)].$$

DUAL-CHERNOFF AXIOM (DCA).

$$\forall S_1, S_2 \in \mathscr{S} : S_1 \subset S_2 \Rightarrow [S_1 \cap C(S_2) = \varnothing \vee S_1 \cap C(S_2) \supset C(S_1)].$$

Another class of important choice-consistency conditions is that of path-independence, due originally to Arrow [3, Chap. VIII, Sect. V] and Plott [12], and various variants thereof. They essentially require that the choice from a set should be independent of the path to be followed *en route* in search for the global choice.

PATH-INDEPENDENCE (PI).

$$\forall S_1, S_2 \in \mathscr{S} : C(S_1 \cup S_2) = C(C(S_1) \cup S_2).$$

WEAK PATH-INDEPENDENCE α (WPI(α)).

$$\forall S_1, S_2 \in \mathscr{S} : C(S_1 \cup S_2) \subset C(C(S_1) \cup S_2).$$

WEAK-PATH-INDEPENDENCE β (WPI(β)).

$$\forall S_1, S_2 \in \mathscr{S} : C(S_1 \cup S_2) \supset C(C(S_1) \cup S_2).$$

Finally we introduce two very weak choice-consistency conditions which still have bites.

SUPERSET AXIOM (SUA).

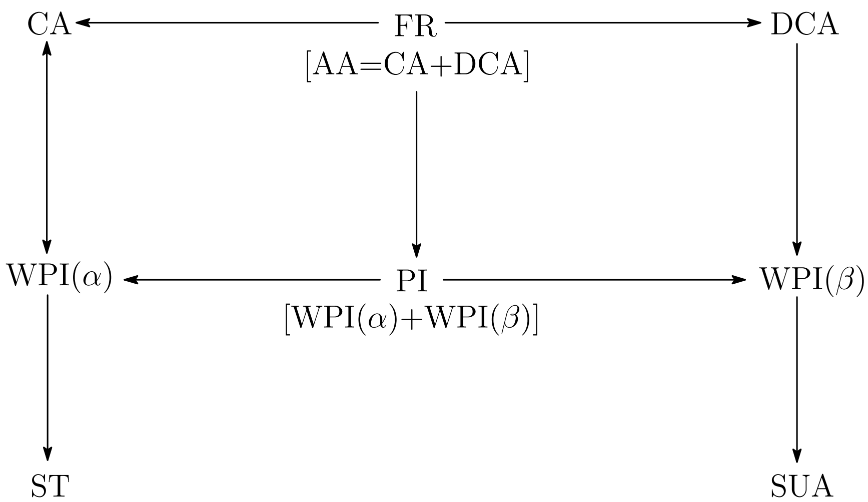
$$\forall S_1, S_2 \in \mathscr{S} : [S_1 \subset S_2 \,\&\, C(S_2) \subset C(S_1)] \Rightarrow C(S_1) = C(S_2).$$

STABILITY AXIOM (ST).

$$\forall S \in \mathscr{S} : C(C(S)) = C(S).$$

To facilitate recollection and later reference, we summarize the logical relationship which holds true among these choice-consistency axioms in the following theorem, where an arrow indicates a logical implication which cannot be reversed in general, while the axioms in square brackets are equivalent to the axiom above them.

THEOREM 1.



Proof. Most of the assertions being either immediate results of the definitions or already established in Blair *et al*. [4], Ferejohn and Grether [6] and Plott [12], we have only to prove that (a) DCA *implies* WPI (β), and (b) WPI(β) *implies* SUA.

(a) Assume that C satisfies the condition DCA on $\mathscr{S}$ and take any $S_1, S_2 \in \mathscr{S}$. Since $C(S_1) \cup S_2 \subset S_1 \cup S_2$ is obviously true, we may invoke DCA to assert that either

$$C(S_1 \cup S_2) \cap [C(S_1) \cup S_2] = \varnothing, \tag{7}$$

or

$$C(S_1 \cup S_2) \cap [C(S_1) \cup S_2] \supset C(C(S_1) \cup S_2) \tag{8}$$

is true. If (8) is indeed the case, we have $C(S_1 \cup S_2) \supset C(C(S_1) \cup S_2)$ and we are home. Assume therefore that (7) is true. Since $S_1 \subset S_1 \cup S_2$ is true, the second use of DCA yields either

$$C(S_1 \cup S_2) \cap S_1 = \varnothing, \tag{9}$$

or

$$C(S_1 \cup S_2) \cap S_1 \supset C(S_1). \tag{10}$$

It follows from (7) that $C(S_1 \cup S_2) \cap C(S_1) = C(S_1 \cup S_2) \cap S_2 = \varnothing$, so that we obtain

$$C(S_1 \cup S_2) \subset S_1 \backslash C(S_1), \tag{11}$$

which negates the validity of (9). Therefore (10) must be true, which however contradicts (11). This concludes our proof of (a).

(b) Assume that C satisfies WPI(β) and let $S_1, S_2 \in \mathscr{S}$ be such that $S_1 \subset S_2$ and $C(S_2) \subset C(S_1)$. Thanks to WPI(β) we then have

$$C(S_2) = C(S_1 \cup S_2) \supset C(C(S_2) \cup S_1) = C(S_1),$$

which, coupled with $C(S_2) \subset C(S_1)$, yields $C(S_1) = C(S_2)$, as desired. ■

Notice that these choice-consistency axioms are properties of a choice function but they may be regarded as properties of a generalized collective choice rule which generates a choice function having the designated properties. With this understanding in mind we will talk about choice-consistency of a GCCR in the following.

3.3. Let us introduce a rather mild unrestricted domain condition on Ψ, which requires that the class of admissible profiles be rich enough to the following extent.

UNRESTRICTED DOMAIN UNDER THE AXIOM OF IDENTITY (UID). *The domain of Ψ consists of all logically possible profiles satisfying the axiom of identity.*

We are now ready to present our first negative theorem on the "fair" GCCRs.

THEOREM 2. *Suppose that there exist at least three social states. Then there exists no GCCR which satisfies UID (Unrestricted Domain under the Axiom of Identity), FE (Fairness Extension), and SUA (Superset Axiom of Choice-Consistency).*

Proof. Take three distinct social states x, y, and z and let $S_1 = \{x, y\}$ and $S_2 = \{x, y, z\}$. Let a profile $\alpha = (\widetilde{R}_1^\alpha, \widetilde{R}_2^\alpha, \ldots, \widetilde{R}_n^\alpha)$ be such that

$$\widetilde{R}_1^\alpha(S_2 \times \{1,2\}) : (x,1), (z,2), (z,1), (y,1), (y,2), (x,2),$$

$$\widetilde{R}_2^\alpha(S_2 \times \{1,2\}) : (z,2), (y,2), (x,2), (x,1), (z,1), (y,1),$$

$$\forall i \in N \backslash \{1,2\} : \widetilde{R}_i^\alpha(S_2 \times \{1,2\}) = \widetilde{R}_1^\alpha(S_2 \times \{1,2\}),$$

where $\widetilde{R}_j^\alpha(S_2 \times \{1,2\})$ denotes the restriction of $\widetilde{R}_j^\alpha$ on $S_2 \times \{1,2\}$ for all $j \in N$, and that

$$\forall (v,j) \in (X \times N) \backslash (S_2 \times \{1,2\}) : \begin{cases} ((x,2),(v,j)) \in P(\widetilde{R}_1^\alpha), \\ ((y,1),(v,j)) \in P(\widetilde{R}_2^\alpha), \\ \forall i \in N \backslash \{1,2\} : ((v,j),(x,1)) \in P(\widetilde{R}_i^\alpha), \end{cases}$$

$$\forall i \in N, \forall (v^1, j^1), (v^2, j^2) \in (X \times N) \backslash (S_2 \times \{1,2\}) :$$
$$((v^1, j^1), (v^2, j^2)) \in I(\widetilde{R}_i^\alpha)$$

It is clearly the case that this profile satisfies the axiom of identity. Notice that $E_q^\alpha(S_2) = \{x, y\}$ and $P(R_N^\alpha) \cap (S_2 \times S_2) = \{(z, y)\}$. Therefore $C^\alpha(S_1) = F^\alpha(S_1) = S_1$ and $C^\alpha(S_2) = F^\alpha(S_2) = \{x\}$, where use is made of the condition FE. We then obtain $S_1 \subset S_2, C^\alpha(S_2) \subset C^\alpha(S_1)$ and $C^\alpha(S_1) \neq C^\alpha(S_2)$. Therefore a GCCR satisfying UID and FE cannot possibly satisfy SUA. ■

3.4. The condition FE demands that the fair state and only the fair state should be chosen when one exists. What if there exists no fair state? To consider this situation, let us define an auxiliary relation R_E^α on X for each $\alpha \in \mathscr{A}$ by

$$R_E^\alpha = \{(x, y) \in X \times X \mid x \in E_q^\alpha(X) \,\&\, y \notin E_q^\alpha(X)\}. \tag{12}$$

Now if we really care about the appeal of equity-as-no-envy as well as Pareto-efficiency, the following requirement may seem to be reasonable, which says basically that a state which is either "more equitable" or "more efficient" than a state which is chosen should itself be among chosen states.

Fairness Inclusion (FI). *If $\alpha \in \mathscr{A}$ and $S \in \mathscr{S}$ are such that $F^\alpha(S) = \varnothing$, then*

(a) $[x \in S, (x, y) \in R_E^\alpha \,\&\, y \in C^\alpha(S)] \Rightarrow x \in C^\alpha(S)$, *and*
(b) $[x \in S, (x, y) \in P(R_N^\alpha) \,\&\, y \in C^\alpha(S)] \Rightarrow x \in C^\alpha(S)$, *where* $C^\alpha = \Psi(\alpha)$.

We also introduce a variant of the Pareto unanimity requirement on a GCCR, but we need careful step forward in this slippery area. To require the exclusion of a state y from a choice set for a binary choice environment $\{x, y\}$ just because x happens to Pareto-dominate y would be grossly inappropriate in the context where we care about equity and the like, since doing so means to empower the Pareto-dominance relation to always outweigh the equity consideration in the binary choice situation. But this lopsided sanctification of the Pareto dominance quite simply contradicts the emphasis put on the equity consideration in the fairness approach. This argument, if accepted, would lead us to the following conditional variant of the Pareto rule.

CONDITIONAL BINARY EXCLUSION PARETO (CBEP). *If an admissible profile α and $x, y \in X$ are such that $E_q^\alpha(\{x, y\}) = \varnothing$ and $(x, y) \in \cap_{i \in N} P(R_i^\alpha)$, then $\{x\} = C^\alpha(\{x, y\})$, where $C^\alpha = \Psi(\alpha)$.*

What emerges out of these mild-looking conditions on a GCCR is another impossibility theorem, which reads as follows.

THEOREM 3. *Suppose that there exist at least three social states. Then there exists no GCCR which satisfies UID (Unrestricted Domain under the Axiom of Identity), FI (Fairness Inclusion), CBEP (Conditional Binary Exclusion Pareto), and CA (Chernoff's Axiom of Choice-Consistency).*

Proof. Take three distinct social states x, y, and z and let $S_1 = \{x, y\}$ and $S_2 = \{x, y, z\}$. Let a profile $\alpha = (\widetilde{R}_1^\alpha, \widetilde{R}_2^\alpha, \ldots, \widetilde{R}_n^\alpha)$ be such that

$$\widetilde{R}_1^\alpha(S_2 \times \{1,2\}) : (x,2),(x,1),(y,2),(y,1),(z,1),(z,2),$$

$$\widetilde{R}_2^\alpha(S_2 \times \{1,2\}) : (x,1),(x,2),(y,1),(y,2),(z,2),(z,1),$$

$$\forall i \in N\backslash\{1,2\} : \widetilde{R}_i^\alpha(S_2 \times \{1,2\}) = \widetilde{R}_1^\alpha(S_2 \times \{1,2\}),$$

where $\widetilde{R}_j^\alpha(S_2 \times \{1,2\}) = \widetilde{R}_j^\alpha \cap [(S_2 \times \{1,2\}) \times (S_2 \times \{1,2\})]$ for all $j \in N$, and that

$$\forall (v,j) \in (X \times N)\backslash(S_2 \times \{1,2\}) : \begin{cases} ((z,2),(v,j)) \in P(\widetilde{R}_1^\alpha), \\ ((z,1),(v,j)) \in P(\widetilde{R}_2^\alpha), \\ \forall i \in N\backslash\{1,2\} : ((v,j),(x,2)) \in P(\widetilde{R}_i^\alpha), \end{cases}$$

$$\forall i \in N, \forall (v^1,j^1),(v^2,j^2) \in (X \times N)\backslash(S_2 \times \{1,2\}) :$$
$$((v^1,j^1),(v^2,j^2)) \in I(\widetilde{R}_i^\alpha)$$

It is easy to verify that this profile satisfies the axiom of identity. Note that $E_q^\alpha(S_2) = \{z\}$ and $P(R_N^\alpha) \cap (S_2 \times S_2) = \{(x,y),(y,z),(x,z)\}$, so that we have $C^\alpha(S_1) = \{x\}$ by virtue of CBEP. Consider now $C^\alpha(S_2)$. If x or y belongs to $C^\alpha(S_2)$, then $z \in C^\alpha(S_2)$ by virtue of FI(a). If $z \in C^\alpha(S_2)$, then x as well as y belongs to $C^\alpha(S_2)$ thanks to FI(b). $C^\alpha(S_2)$ being non-empty, we should then conclude that $C^\alpha(S_2) = S_2$. Then we have $S_1 \subset S_2$, $S_1 \cap C^\alpha(S_2) = \{x,y\} \not\subset C^\alpha(S_1) = \{x\}$, which implies that a GCCR satisfying UID, FI and CBEP cannot possibly satisfy CA. ■

3.5. Several remarks seem to be in order here.

Firstly let us point out that there exists a concrete example of fairness-extending as well as fairness-including GCCR which is due essentially to Goldman and Sussangkarn [8]. For any profile $\alpha \in \mathscr{A}$ and any $S \in \mathscr{S}$, let a binary relation R^α on X be defined by $R^\alpha = P(R_N^\alpha) \cup R_E^\alpha$ and let $R^\alpha(S)$ stand for the restriction of R^α on $S : R^\alpha(S) = R^\alpha \cap (S \times S)$. We then define a choice set $C_{GS}^\alpha(S)$ for S by

$$C_{GS}^\alpha(S) = \{x \in S \mid \forall y \in S : (x,y) \in T(R^\alpha(S)) \vee (y,x) \notin T(R^\alpha(S))\}, \quad (13)$$

where $T(R^\alpha(S))$ denotes the transitive closure of $R^\alpha(S)$. Associating the well-defined choice function C_{GS}^α on $\mathscr{S}$ thereby constructed with the profile α we have started from, we have a complete description of a GCCR $\Psi_{GS} : \Psi_{GS}(\alpha) = C_{GS}^\alpha$ for all $\alpha \in \mathscr{A}$. It is easy, if tedious, to show that Ψ_{GS} satisfies the condition FE as well as the condition FI for all $\alpha \in \mathscr{A}$. Thanks to Theorem 2 we can assert without further ado that Ψ_{GS} cannot possibly satisfy the superset axiom of choice-consistency. We may also prove that Ψ_{GS} fails to satisfy Chernoff's axiom of choice-consistency as well, but it does satisfy the stability axiom, although the latter property may presumably be too weak to celebrate Ψ_{GS} for its success in this arena. The power of Theorem 2 is such that it asserts by one stroke that we cannot possibly improve the performance of Ψ_{GS} unless we renounce the wide applicability of our GCCR or the nice choice-consistency property thereof.

Secondly we may assert the following simple corollaries of Theorems 1, 2, and 3. In view of the strong intuitive appeal of path-independence argument, these corollaries may better crystallize the logical difficulty identified by Theorems 2 and 3.

Corollary 1. *Suppose that there exist at least three social states. Then there exists no GCCR which satisfies UID (Unrestricted Domain under the Axiom of Identity), FE (Fairness Extension) and WPI*(β)*(Weak Path-Independence* β*).*

Corollary 2. *Suppose that there exist at least three social states. Then there exists no GCCR which satisfies UID (Unrestricted Domain under the Axiom of Identity), FI (Fairness Inclusion), CBEP (Conditional Binary Exclusion Pareto) and WPI*(α) *(Weak Path-Independence* α*).*

Thirdly we should note that Theorem 2 as well as Theorem 3 does not invoke any interprofile independence condition, which has often been nominated as *the* culprit of Arrovian impossibility theorems. Indeed, only a single profile is made effective use of in proving Theorems 2 and 3, respectively, so that the *profiles richness condition* UID is in fact much stronger than is needed.[8] Instead of requiring UID we may do throughout with the following *states richness condition* suggested by Pollak [13] and Roberts [15], which is the single-profile analogue of the multiple-profile requirement UID. Let $\alpha \in \mathscr{A}$ denote the given fixed profile.

States Richness Condition (SRC). *Let* $\beta(S_0)$ *denote any logically possible sub-profile over the hypothetical triple set* S_0*. Then there exists a one-to-one correspondence* γ_β *from* S_0 *into* X *such that*

$$((\gamma_\beta(x), i),\ (\gamma_\beta(y), j)) \in \widetilde{R}_k^\alpha \leftrightarrow ((x, i), (y, j)) \in \widetilde{R}_k^\beta$$

for all $x, y \in S_0$ *for all* $i, j, k \in N$.[9]

It should be clear that the condition SRC, which essentially requires that the set of states X is rich enough, may replace the condition UID, which is the requirement to the effect that the set of profiles $\mathscr{A}$ is rich enough, to generate single-profile analogue of Theorems 2 and 3 and Corollaries 1 and 2.

4 Concluding Remarks

It is hoped that our results reported in this paper, which are largely negative, will help clarify the nature and potentiality of the fairness-as-no-envy approach in the theory of fairness and justice. In concluding this chapter, a few remarks are due.

(a) According to Varian [20, p.65], "[s]ocial decision theory asks for too much out of the [preference aggregation] process in that it asks for an entire *ordering* of the various

[8] In this respect, our Theorem 2 and Theorem 3 are similar in nature to Sen's [16, Chap. 6*] *impossibility of a Paretian liberal*.

[9] More explicitly, $\beta(S_0) = (\widetilde{R}_1^\beta(S_0 \times N), \widetilde{R}_2^\beta(S_0 \times N), \ldots, \widetilde{R}_n^\beta(S_0 \times N))$.

social states... . The original question asked only for a good allocation; there was no requirement to rank all allocations. The fairness criterion in fact limits itself to answering the original question. It is limited in that it gives no indication of the merits of two nonfair allocations, but by restricting itself in this way it allows for a reasonable solution to the original problem." This contrast between "social decision theory" and "fairness criterion" is no doubt a useful one, but it seems to us that the two approaches may well be subsumed in a more general choice-functional collective choice framework. In doing so, we may enrich our understanding of one theory in the light of the implications of the other theory on the common ground and *vice versa*. This is precisely the kind of exercise we tried to perform in this chapter.

(b) It is often suggested that the prime virtue of the theory of fairness is that it requires no such things as *externally imposed* interpersonal welfare comparisons, hypothetical welfare functions, or fictitious original position. Notice that our analysis of the concept of fairness in the framework of social choice theory fully retains this alleged prime virtue of the theory of fairness. One may even claim, following Alchian [1], that what is involved in our GCCR framework is not an interpersonal welfare comparison but an intrapersonal, intersituational comparison.

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Chapter 6
The Informational Basis of the Theory of Fair Allocation*

1 Introduction

The theory of fair allocation studies allocation rules which select, for every economy in a given class, a subset of feasible allocations on the basis of efficiency and fairness properties. It was initiated by Foley [14], Kolm [16] and Varian [31] among others, who focussed on the concept of no-envy. Since then it has been extended to cover many other notions of fairness and a great variety of economic contexts (production, public goods, etc.) by many authors.[1] This theory contains some negative results, because it is usually impossible to find solutions which satisfy all conceivable requirements of efficiency and equity simultaneously, but its hallmark is a richness of positive results. By now, not only are there many interesting *allocation rules* uncovered in the literature, but also they are fully characterized as the only rules satisfying some sets of reasonable axioms.

Compared to the theory of social choice, this makes a great contrast. In social choice theory, Arrow's impossibility theorem has been shown to remain valid in most economic or abstract contexts. This theorem, like all the theory of social choice, is about *social preferences* which rank all options in a given set on the basis of individual preferences over these options. The theorem states that there is no way to construct social preferences as a function of individual preferences if this function is required to satisfy basic principles of unanimity (Weak Pareto: if everybody prefers x to y, so does society), impartiality (Non-Dictatorship: no individual can always impose his strict preferences) and informational parsimony (Independence of Irrelevant Alternatives: social preferences over any subset of alternatives depend only on individual preferences over this subset).

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[1]For a survey, see Moulin and Thomson [19].

Impossibilities in social choice theory, possibilities in fair allocation theory — this contrast requires an explanation. Such an explanation is not only interesting from a purely theoretical viewpoint, but bears on the ability of social choice theorists to provide helpful concepts and tools to public economists and decision-makers. The possibility of social choice is at stake.

The starting point of this chapter is a simple observation. The main explanation which can be found in the literature is not satisfactory. It says that the theory of social choice seeks full-fledged orderings of all alternatives, whereas the theory of fair allocation is satisfied with a selection of efficient and equitable allocations in the feasible set. In other words, Arrow's impossibility theorem applies to *social preferences* and does not apply to *allocation rules*. The problem with this explanation is that an allocation rule is formally equivalent to an ordering of all allocations. It is a coarse ordering, with only two indifference classes, the "good" and the "bad". But a coarse ordering is an ordering. And there is nothing in Arrow's framework which requires social preferences to have more than two indifference classes. Therefore Arrow's theorem does apply to allocation rules, and the theory of fair allocation is essentially a part of the theory of social choice.

This simple point raises a question and suggests an answer. The question is: What, then, is the explanation for the possibility results in fair allocation theory? The answer is: Allocation rules must violate some of the axioms of Arrow's theorem. It is indeed a simple exercise to show that the prominent solutions in the theory of fair allocation, viewed as social preferences, do violate the axioms of Weak Pareto and of Independence of Irrelevant Alternatives. It is even quite easy to see why Weak Pareto is violated by allocation rules. This axiom requires strict social preference in favor of any Pareto-dominating allocation, and since there are many (chained) instances of Pareto-domination in relevant domains, it entails that social preferences must have more than two indifference classes. Allocation rules can at best satisfy the Pareto-efficiency condition (requiring the selected allocations not to be Pareto-dominated).

In this chapter, we generalize these observations and show why all reasonable (i.e. minimally impartial) allocation rules in fair allocation theory must also violate the axiom of Independence of Irrelevant Alternatives. Moreover, we examine how much weakening of this axiom (i.e., introducing more information about individual preferences) is required for a reasonable allocation rule to be possible. In particular, we find that less weakening of Independence of Irrelevant Alternatives is needed for allocation rules, thanks to the fact that Weak Pareto is weakened into the Pareto efficiency condition (so as to be compatible with social preferences having two indifference classes only), than for social preferences satisfying Weak Pareto. In other words, there is a trade-off between Pareto conditions and independence conditions.

In summary, the main lessons of this chapter are the following. First, *the theory of fair allocation succeeds in obtaining possibility results mainly because it relaxes the axiom of Independence of Irrelevant Alternatives.* Second, in order to obtain possibility results, *the theory of social choice (with social preferences satisfying Weak Pareto) needs more information about individual preferences (more weakening of Independence of Irrelevant Alternatives) than the theory of fair allocation (with coarse social preferences satisfying* *only the Pareto-efficiency condition).*

This chapter is organized as follows. Section 2 briefly examines the explanation of the possibility results in fair allocation theory as given by the literature. This examination seems in order because, in all fairness, the literature is not as simple-minded as the above summary suggests. Section 3 presents some simple examples in order to show how allocation rules can be viewed as social preferences, and what axioms of Arrow's theorem they respect or violate. Then Section 4 introduces the model and the main formal notions. The reason why allocation rules must violate Independence of Irrelevant Alternatives is studied in Section 5. In Section 6 we then discuss some weak variants of Independence of Irrelevant Alternatives, and examine how such variants can be satisfied by allocation rules. Section 7 compares the informational basis of the theory of fair allocation to that of social choice theory (with Paretian social preferences), delineating the trade-off between Pareto conditions and independence conditions. In Section 8 we come back on the above assertion that the theory of fair allocation is just a part of the theory of social choice, and discuss various possible unifications of these theories, and their relative merits. Section 9 concludes.

2 Explanations from the Literature

Most authors have stressed two differences between the two theories. The one most often mentioned is about preferences versus selection. Varian [31] argues as follows:

> 'Social [choice] theory asks for too much out of the process in that it asks for an entire *ordering* of the various social states (allocations in this case). The original question asked only for a "good" allocation; there was no requirement to rank all allocations. The fairness criterion in fact limits itself to answering the original question. It is limited in that it gives no indication of the merits of two nonfair allocations, but by restricting itself in this way it allows for a reasonable solution to the original problem.' (p. 65)

Similarly, Kolm [17] is ironic about social preferences:

> 'The requirement of a social ordering is indeed problematic at first sight: Why would we want to know the 193th best alternative? Only the first best is required for the choice.' (p. 439)

In his famous survey on social choice theory, Sen [25] also emphasizes this contrast:

> 'The specified subset is seen as good, but there is no claim that they represent the "best" alternatives, all *equally* choosable. There is no attempt to give an answer to the overall problem of social choice, and the exercise is quite different from the specification of a social preference over X.' (p. 1106)

And most recently, Moulin and Thomson [19] have compared the two theories in these terms:

> 'In social choice theory, the focus is commonly on obtaining a complete ranking of the set of feasible alternatives as a function of the profile of individual preferences. (...) Consider now the axiomatic investigations of resource allocation. As their counterparts in the theory of cooperative games, their focus is on the search for allocation rules, no attempt being made at obtaining a complete ranking of the entire feasible set.' (p. 104)

The second difference noticed by these authors is that economic models enable the analyst to take account of the structure of allocations. Varian mentions only the fact that the theory of fair allocation can focus on self-centered preferences (individuals being interested only in their own consumption), while Sen has written about the fairness literature:

> 'First, it has shown the relevance of informational parameters that the traditional social choice approaches have tended to ignore in the single-minded concern with individual orderings of complete social states. Comparisons of different persons' positions within a state have been brought into the calculation, enlarging the informational basis of social judgments. Second, in raising rather concrete questions regarding states of affairs, the fairness literature has pushed social choice theory in the direction of more structure.' (p. 1111)

Similarly, Moulin and Thomson have argued that

> 'the models of resource allocation take full account of the microeconomic structure of the problems to be solved. (...) This descriptive richness permits a great deal of flexibility at two levels. First, properties of allocation rules can be formulated directly in terms of the physical attributes of the economy (...). Second, the rich mathematical structure of microeconomic models gives rise to a host of variations on each general principle.' (p. 105)

However, this second difference is about additional requirements formulated in a richer framework, and can hardly explain the relative success of the theory of fair allocation. This was noted by Moulin and Thomson, who have concluded:

> 'Note that social choice theory itself has recently developed in a similar direction, widening its framework by incorporating information about economic environments (...). But as its objective has remained to obtain complete rankings of sets of feasible alternatives, its conclusions have so far remained largely negative.' (ibid.)

Actually, Arrow's initial presentation of his theorem (Arrow [1, 2]) was already formulated in an economic setting, with self-centered preferences. He indeed considered the possibility that a more concrete framework, with a domain restricted to standard consumer preferences, might alter the general outlook of social choice, and he concluded negatively. This conclusion has been fully confirmed by the more recent research alluded to by Moulin and Thomson.

All in all, one can safely conclude that the common explanation for the possibility results in the theory of fair allocation is that it does not seek a full-fledged ordering.

3 Allocation Rules as Social Preferences

As explained in the introduction, an allocation rule, in effect, splits the set of allocations in two parts, the good and the bad. Even though the intention of particular authors in this field may not have been to give an ordering of allocations, this twofold partition is, *formally*, an ordering. Now, in view of the above quotations from the literature, one may wonder whether the best interpretation of allocation rules is to view them as partial orderings (quasi-orderings) or as complete orderings. An allocation rule may be viewed as a partial ordering if the good allocations are deemed non-comparable, and similarly for the bad ones, as suggested above by Varian and Sen. But this would not save the thesis that Arrow's theorem does not apply to allocation rules. Firstly, the social choice literature has extended the bulk of Arrow's theorem to quasi-orderings.[2] Secondly, nothing prevents one from deriving a complete ordering from any allocation rule. This means that the theory of fair allocation is, willy nilly, able to provide complete orderings, and the puzzle of this success, contrasted to Arrow's impossibility, remains.

In this section we examine some examples in order to provide more intuition about how allocation rules can be viewed as social preferences, and as such may be submitted to the test of Arrow's axioms. Let us for the moment consider a simple Edgeworth box setting with two goods and two individuals. Every individual has self-centered preferences about his own bundles of these two goods. The set of feasible alternatives contains all allocations for which total consumption does not exceed a fixed available amount of the two goods. From individual self-centered preferences over bundles one can derive individual preferences over allocations, simply by considering that an individual prefers an allocation to another whenever he prefers his own bundle in this allocation to his own bundle in the other allocation. Therefore we are essentially in a particular version of Arrow's framework, with individual preferences over a given set of allocations and the question is whether one can derive social preferences over this same set of allocations from any profile of individual preferences.

Arrow's theorem does apply to such a simple setting, as shown by Bordes, Campbell and Le Breton [4]. More precisely, they assume that the domain of individual preferences contains all continuous, strictly monotonic and strictly convex preference relations over bundles, and study social ordering functions defining a complete ordering for any profile of preferences. They show that any such function satisfying Weak Pareto and Independence of Irrelevant Alternatives must be such that one particular individual imposes her own strict preferences over all interior allocations.[3]

A prominent solution from the theory of fair allocation, in this simple setting, is the Egalitarian Walrasian allocation rule, which selects the competitive equilibrium alloca-

[2] In particular, Weymark [33] has studied the application of Arrow's axioms to partial orderings, and obtained oligarchy results. More interestingly, by adding anonymity to the axioms, he characterized the Pareto partial ordering. Although his results are obtained in an abstract framework with unrestricted preferences, they strongly suggest that little can be gained by abandoning completeness.

[3] This is not a full dictator, since this individual is not able to impose her strict preferences over all allocations. But this result is sufficiently "dictatorial" to be interpreted as preserving the bulk of Arrow's theorem.

tions with equal budgets. (One may describe it as first dividing all available resources equally among individuals, and then letting them trade in a competitive market.) This allocation rule defines simple two-tier social preferences, such that any equal-budget competitive equilibrium is ranked above any other type of allocation, all equal-budget competitive equilibria are socially indifferent, and all other allocations are socially indifferent. This is a complete ordering of all the allocations of the relevant set.

Since such an ordering is defined for every profile of individual preferences in the above domain, one obtains a social ordering *function* which satisfies all requirements of Arrow's framework. But it does not satisfy all axioms of Arrow's theorem, and to this we now turn.

This social ordering function does not satisfy Weak Pareto, for an obvious reason already explained above: All non-selected allocations are deemed socially indifferent, in spite of the fact that some of them Pareto-dominate others. Nonetheless, a weaker Pareto condition is satisfied, since the preferred allocations are never Pareto-dominated by the other allocations. Pareto-efficiency of the selected allocations is indeed the relevant condition for allocation rules. But it is important to reckon that this is weaker than Weak Pareto.

More interestingly, this social ordering function does not satisfy Independence of Irrelevant Alternatives. This is illustrated on Figure 1, which features two allocations, x and y. Two different profiles are shown on panel (a) and panel (b). In both profiles, the first individual prefers allocation y (since she receives more of both goods in y), and the other individual has the opposite preferences. By Independence of Irrelevant Alternatives, the fact that individual preferences about x and y are the same in the two profiles implies that social preferences about the two allocations should be identical for the two cases. But this is not satisfied with the social preferences derived from the Egalitarian Walrasian allocation rule. Indeed, in panel (a), x is an equal-budget competitive equilibrium allocation and y is not, so that x is socially preferred to y. The reverse occurs in panel (b).

It remains to check that there is no dictator with such social preferences. This is again illustrated in Figure 1, since social preferences go against the first individual's preferences on panel (a) and against the other's preferences on panel (b). It is actually easy to generalize from this example, and see that an allocation rule cannot have a dictator in Bordes, Campbell and Le Breton's sense. Indeed, when individual preferences are strictly monotonic a dictator has fine-grained preferences, which cannot be obeyed by a coarse social ordering with only two indifference classes. Therefore, an allocation rule, viewed as social preferences, always trivially satisfies the Non-Dictatorship condition of Arrow's theorem.

Let us now consider a second example. Another prominent solution in fair allocation theory is Pazner and Schmeidler's [22] Egalitarian-Equivalent allocation rule. This allocation rule selects the Pareto-efficient allocations such that every individual is indifferent between her own bundle and a common reference bundle which is proportional to the

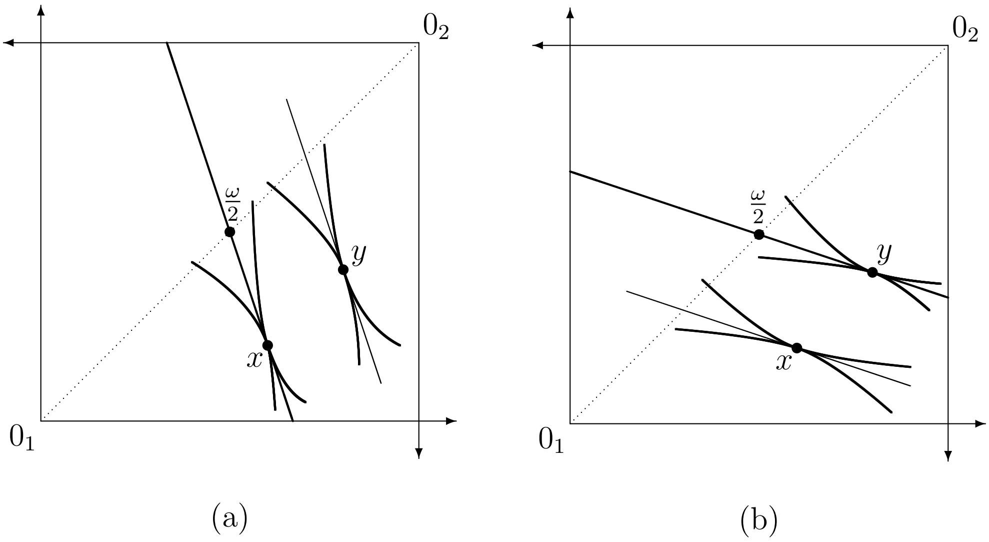

Figure 1: The Egalitarian Walrasian Allocation Rule Violates IIA

total available resources. Similarly as above, one can derive two-tier social preferences from this allocation rule. Again, it does not satisfy Weak Pareto, and does satisfy the Non-Dictatorship condition, for the same obvious reasons as above. The fact that it does not satisfy Independence of Irrelevant Alternatives is illustrated in Figure 2. In panel (a), allocation x is egalitarian-equivalent and allocation y is not, while the reverse holds on panel (b). By Independence of Irrelevant Alternatives, however, social preferences should be the same in the two cases, since individual preferences about x and y are identical.

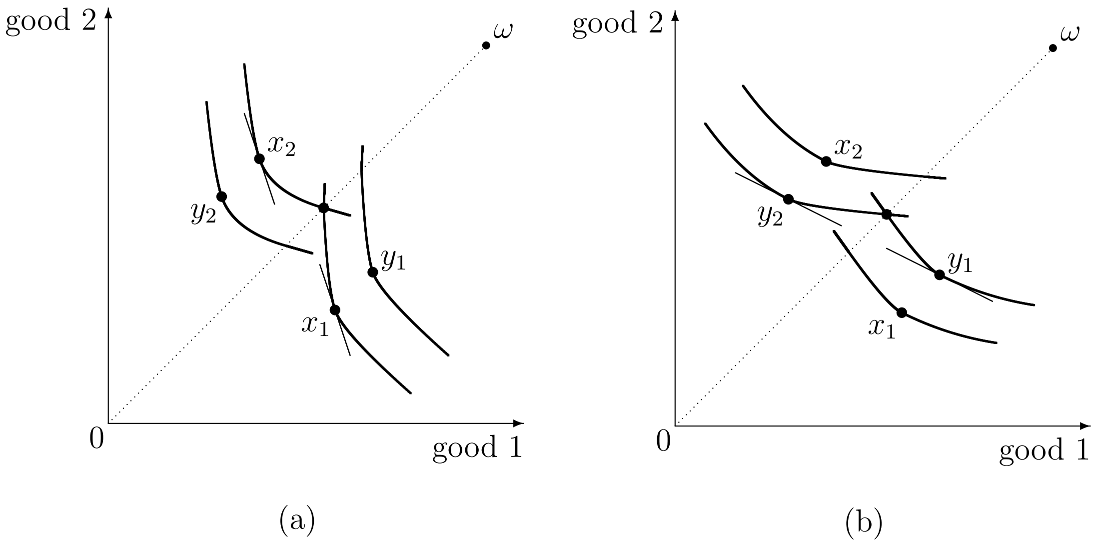

Figure 2: The Egalitarian-Equivalent Allocation Rule Violates IIA

Figures 1 and 2 provide intuition for the reason why it is unlikely that an allocation rule, viewed as a social ordering function, will satisfy Independence of Irrelevant Alternatives. In these two examples, information about whether any individual prefers x or y is not sufficient to judge how good the allocations are. In the case of the Egalitarian Walrasian allocation rule, one needs to know at least the marginal rates of substitution at the relevant bundles. In the case of the Egalitarian-Equivalent allocation rule, one needs to know the intersection of the relevant indifference curves with a particular ray in the space of goods. The next sections study how stringent, in all generality, the Independence condition is for allocation rules viewed as social ordering functions.

4 Model and Definitions

Before going to technicalities, let us take stock and see what remains to be clarified. The previous sections have established the following: 1) the standard explanation for the possibility results in fair allocation, in view of Arrow's impossibility theorem, is that allocation rules are not orderings, so that Arrow' theorem does not apply to them; 2) actually, allocation rules do provide complete orderings, so that Arrow's theorem does apply to them; the correct explanation must be that allocation rules violate some of Arrow's axioms; 3) allocation rules, due to the fact that they yield coarse (two-tier) social preferences, always violate Weak Pareto and satisfy Non-Dictatorship; 4) prominent examples of allocation rules violate Independence of Irrelevant Alternatives.

What remains to be seen, at this stage, is whether any reasonable allocation rule may satisfy Independence of Irrelevant Alternatives. What we have to do, therefore, is to examine the implications, for an allocation rule (viewed as two-tier social preferences), of satisfying this axiom. For this we must formally define the concept of an allocation rule, the concept of social preferences, and, more importantly, the concept of two-tier social preferences associated to an allocation rule, so that Arrow's axioms may be correctly applied to an allocation rule.

4.1 The Model

The model adopted here is just an immediate extension of the simple framework of the previous section.

The *population* is fixed. Let $N = \{1, ..., n\}$ be the set of *agents* where $2 \leq n < \infty$. There are ℓ *goods* indexed by $k = 1, ..., \ell$ where $2 \leq \ell < \infty$. Agent i's *consumption bundle* is a vector $x_i = (x_{i1}, ..., x_{i\ell}) \in \mathbb{R}^{\ell}_{+}$. An *allocation* is denoted $x = (x_1, ..., x_n) \in \mathbb{R}^{n\ell}_{+}$.

A *preordering* is a reflexive and transitive binary relation. Agent i's *preferences* are described by a complete preordering R_i (strict preference P_i, indifference I_i) on $\mathbb{R}^{\ell}_{+}$. A *profile of preferences* is denoted $\mathbf{R} = (R_1, ..., R_n)$. Let $\mathcal{R}$ be the set of continuous, convex, and strictly monotonic preferences over $\mathbb{R}^{\ell}_{+}$.

Let π be a bijection on N. For each $x \in \mathbb{R}^{n\ell}_{+}$, define $\pi(x) = (x'_1, ..., x'_n) \in \mathbb{R}^{n\ell}_{+}$ by $x'_i = x_{\pi(i)}$ for all $i \in N$, and for each $\mathbf{R} \in \mathcal{R}^n$, define $\pi(\mathbf{R}) = (R'_1, ..., R'_n) \in \mathcal{R}^n$ by $R'_i = R_{\pi(i)}$ for all $i \in N$. Let Π be the set of all bijections on N.

There is no production in our model, and the amount of *total resources* is fixed and represented by the vector $\omega \in \mathbb{R}^{\ell}_{++}$. An allocation $x \in \mathbb{R}^{n\ell}_{+}$ is *feasible* if $\sum_{i \in N} x_i \leq \omega$.[4] Let F be the set of all feasible allocations.

For each $\mathbf{R} \in \mathcal{R}^n$, let $E(\mathbf{R})$ denote the set of *Pareto-efficient allocations*. Because of strict monotonicity of preferences, there is no need to distinguish Pareto-efficiency in the strong sense and in the weak sense.

A *social ordering function* (SOF) is a function $\bar{R}$ defined on $\mathcal{R}^n$, such that for all $\mathbf{R} \in \mathcal{R}^n$, $\bar{R}(\mathbf{R})$ is a complete preordering on the set of allocations F. Let $\bar{P}(\mathbf{R})$ (resp. $\bar{I}(\mathbf{R})$) denote the strict preference (resp. indifference) relation derived from $\bar{R}(\mathbf{R})$.

An *allocation rule* (AR) is a set-valued mapping S defined on $\mathcal{R}^n$, such that[5] for all $\mathbf{R} \in \mathcal{R}^n$, $S(\mathbf{R})$ is a non-empty subset of F. An AR S is *essentially single-valued* if all selected allocations are Pareto-indifferent:

$$\forall x, y \in S(\mathbf{R}), \forall i \in N,\ x_i\ I_i\ y_i.$$

We may now provide precise definitions for the two ARs informally introduced in the previous section. The first one is the **Egalitarian Walrasian AR** S_W, defined as follows: $x \in S_W(\mathbf{R})$ if $x \in F$ and there is $p \in \mathbb{R}^{\ell}_{++}$ such that for all $i \in N$,

$$\forall y \in \mathbb{R}^{\ell}_{+},\ p \cdot y \leq p \cdot \frac{\omega}{n} \Rightarrow x_i\ R_i\ y.$$

The second allocation rule is the **Pazner-Schmeidler AR** S_{PS}, defined by: $x \in S_{PS}(\mathbf{R})$ if $x \in E(\mathbf{R})$ and there is $\alpha \in \mathbb{R}_{+}$ such that for all $i \in N$,

$$x_i\ I_i\ \alpha\omega.$$

With each AR S one can associate the (two-tier) SOF $\bar{R}_S$ defined as follows: for all $\mathbf{R} \in \mathcal{R}^n$, and all $x, y \in F$,

$$x\ \bar{R}_S(\mathbf{R})\ y \Leftrightarrow x \in S(\mathbf{R}) \text{ or } y \notin S(\mathbf{R}).$$

One then has: for all $\mathbf{R} \in \mathcal{R}^n$, and all $x, y \in F$,

$$x\ \bar{P}_S(\mathbf{R})\ y \Leftrightarrow \text{not}[y\ \bar{R}_S(\mathbf{R})\ x] \Leftrightarrow x \in S(\mathbf{R}) \text{ and } y \notin S(\mathbf{R}).$$

Conversely, with each SOF $\bar{R}$ one can associate the AR $S_{\bar{R}}$ defined as follows: for all $\mathbf{R} \in \mathcal{R}^n$,

$$S_{\bar{R}}(\mathbf{R}) = \{x \in F \mid \forall y \in F,\ x\ \bar{R}(\mathbf{R})\ y\}.$$

[4] Vector inequalities are denoted as usual: $\geq, >$, and $\gg$.

[5] An alternative definition of SOFs and ARs makes them a function of ω as well as $\mathbf{R}$. This is useful when changes in ω are studied, but here we focus only on the information about preferences, and since ω is kept fixed throughout the chapter, we omit this argument.

Notice that for each AR S,

$$S_{\bar{R}_S} = S,^{6} \tag{0}$$

and for each SOF $\bar{R}$ that has at most two indifference classes,

$$\bar{R}_{S_{\bar{R}}} = \bar{R}.^{7} \tag{0}$$

Hence, there exists a precise one-to-one correspondence between the class of all ARs and the class of all SOFs that have at most two indifference classes. Figure 3 illustrates this correspondence.

$$S \; \underset{S_{\bar{R}}}{\overset{\bar{R}_S}{\rightleftarrows}} \; \bar{R}$$

Figure 3: One-to-One Correspondence between ARs and SOFs with at most Two Indifference Classes

What we want to study here is the application of Arrow's axioms to a particular class of SOFs, namely, the SOFs that are associated to allocation rules. It is therefore convenient to give them a special name. Let an "*ARSOF*" be a SOF that has at most two indifference classes, and is therefore associated to an AR. Formally, an ARSOF is a SOF $\bar{R}$ for which there exists an AR S such that $\bar{R} = \bar{R}_S$.

We will say that an ARSOF is *essentially single-valued* if its associated AR is essentially single-valued.

4.2 Arrow's Axioms

We are now ready to give precise definitions of Arrow's three conditions.

Weak Pareto: $\forall \mathbf{R} \in \mathcal{R}^n, \forall x, y \in F$, if $\forall i \in N,\ x_i P_i y_i$, then $x \bar{P}(\mathbf{R}) y$.

---

[6] One has, for all $\mathbf{R} \in \mathcal{R}^n$:

$$\begin{aligned} S_{\bar{R}_S}(\mathbf{R}) &= \{x \in F \mid \forall y \in F,\ x\ \bar{R}_S(\mathbf{R})\ y\} \\ &= \{x \in F \mid \forall y \in F,\ x \in S(\mathbf{R}) \text{ or } y \notin S(\mathbf{R})\} \\ &= \{x \in F \mid \forall y \in S(\mathbf{R}),\ x \in S(\mathbf{R})\} \\ &= \{x \in F \mid x \in S(\mathbf{R})\} \\ &= S(\mathbf{R}). \end{aligned}$$

[7] One has, for all $\mathbf{R} \in \mathcal{R}^n$ and all $x, y \in F$:

$$\begin{aligned} x\ \bar{R}_{S_{\bar{R}}}(\mathbf{R})\ y &\Leftrightarrow x \in S_{\bar{R}}(\mathbf{R}) \text{ or } y \notin S_{\bar{R}}(\mathbf{R}) \\ &\Leftrightarrow [\forall z \in F,\ x\ \bar{R}(\mathbf{R})\ z] \text{ or } [\exists z \in F,\ z\ \bar{P}(\mathbf{R})\ y]. \end{aligned}$$

In the former case, we obtain $x\ \bar{R}(\mathbf{R})\ y$ by choosing $z = y$. In the latter case, y belongs to the lower indifference class of $\bar{R}(\mathbf{R})$, so that we have $x\ \bar{R}(\mathbf{R})\ y$ irrespective of whether x belongs to the higher or lower indifference class.

Independence of Irrelevant Alternatives (IIA): $\forall \mathbf{R}, \mathbf{R}' \in \mathcal{R}^n, \forall x, y \in F$, if $\forall i \in N$,

$$\begin{array}{ccc} x_i\ R_i\ y_i & \Leftrightarrow & x_i\ R'_i\ y_i \\ y_i\ R_i\ x_i & \Leftrightarrow & y_i\ R'_i\ x_i, \end{array}$$

then $x\ \bar{R}(\mathbf{R})\ y \Leftrightarrow x\ \bar{R}(\mathbf{R}')\ y$.

In economic domains, it is common to refine the definition of non-dictatorship so as to allow for slight strengthenings of the usual axiom. Let $X \subset F$ be given.[8]

Non-Dictatorship (over X): There does not exist $i_0 \in N$ such that:

$$\forall \mathbf{R} \in \mathcal{R}^n, \forall x, y \in X, x_{i_0}\ P_{i_0}\ y_{i_0} \Rightarrow x\ \bar{P}(\mathbf{R})\ y.$$

In addition to these axioms, it will be useful to refer to a full anonymity condition, which is stronger than Non-Dictatorship but quite appealing on grounds of impartiality:[9]

Anonymity: $\forall \mathbf{R} \in \mathcal{R}^n, \forall x, y \in F, \forall \pi \in \Pi$,

$$x\ \bar{R}(\mathbf{R})\ y \Leftrightarrow \pi(x)\ \bar{R}(\pi(\mathbf{R}))\ \pi(y).$$

4.3 Arrow's Axioms and Allocation Rules

Our purpose is to examine the implications of Arrow's axioms for the allocation rules of fair allocation theory. Arrow's axioms, as defined above for SOFs, can be applied directly to ARSOFs, which are just a particular kind of SOFs. But it is quite illuminating to see the exact constraints the axioms impose on the AR associated to an ARSOF. That is, one can directly rewrite Arrow's axioms in terms of the constraints they impose on an AR. This is just a simple exercise in substituting definitions, but it appears quite useful for the intuition. In addition, the obtained formulations are helpful when one comes to think about weakening the axioms, which will be an important topic in this chapter.

When $\bar{R}$ is an ARSOF, $x\ \bar{R}(\mathbf{R})\ y$ is logically equivalent to $[x \in S_{\bar{R}}(\mathbf{R})$ or $y \notin S_{\bar{R}}(\mathbf{R})]$, and $x\ \bar{P}(\mathbf{R})\ y$ is equivalent to $[x \in S_{\bar{R}}(\mathbf{R})$ and $y \notin S_{\bar{R}}(\mathbf{R})]$. Substituting these expressions, we obtain the following. The direct translation of Weak Pareto yields:

Weak Pareto (for ARSOF): $\forall \mathbf{R} \in \mathcal{R}^n, \forall x, y \in F$, if $\forall i \in N,\ x_i\ P_i\ y_i$, then $x \in S_{\bar{R}}(\mathbf{R})$ and $y \notin S_{\bar{R}}(\mathbf{R})$.

Formulated in this way, it is immediate that Weak Pareto is too strong for ARSOFs. If one takes $x, y, z \in F$ such that for all i, $x_i\ P_i\ y_i\ P_i\ z_i$, the axiom requires $y \in S_{\bar{R}}(\mathbf{R})$

[8]For instance, as explained above, the relevant set for the Non-Dictatorship condition may be the set of interior allocations.

[9]Notice that the standard "public good" anonymity condition (stating that social preferences should be invariant to permutations of individual preferences over *allocations*) would not make sense in the current "private good" setting, since individual i's preferences focus on his own bundle. A permutation of preferences over *allocations* would mean that he would focus on another individual's bundle, which is not permitted in the domain.

and also $y \notin S_{\bar{R}}(\mathbf{R})$, a contradiction. This was already explained in the introduction. The standard weakening of this axiom, for applications to ARs, is the following:

Pareto-Efficiency (for ARSOF): $\forall \mathbf{R} \in \mathcal{R}^n$, $\forall x, y \in F$, if $\forall i \in N$, $x_i \; P_i \; y_i$, then $y \notin S_{\bar{R}}(\mathbf{R})$.

We are actually more familiar with the following, equivalent, formulation:

Pareto-Efficiency (for ARSOF): $\forall \mathbf{R} \in \mathcal{R}^n$, $S_{\bar{R}}(\mathbf{R}) \subseteq E(\mathbf{R})$.

Interestingly, however, this is not the only conceivable weakening of Weak Pareto for ARSOFs. Another sensible condition, which is logically weaker than Pareto-Efficiency, is the following:

Partial Pareto (for ARSOF): $\forall \mathbf{R} \in \mathcal{R}^n$, $\forall x, y \in F$, if $\forall i \in N$, $x_i \; P_i \; y_i$, then $x \in S_{\bar{R}}(\mathbf{R})$ or $y \notin S_{\bar{R}}(\mathbf{R})$.

This is equivalent to the following condition: $\forall \mathbf{R} \in \mathcal{R}^n$, $\forall x, y \in F$, if $\forall i \in N$, $x_i \; P_i \; y_i$, then $y \in S_{\bar{R}}(\mathbf{R}) \Rightarrow x \in S_{\bar{R}}(\mathbf{R})$. If an ARSOF $\bar{R}$ satisfies this condition, then for each $\mathbf{R} \in \mathcal{R}^n$, there exists a subset $T \subseteq F$ such that

$$S_{\bar{R}}(\mathbf{R}) \supseteq \bigcup_{y \in T} [\{y\} \cup \{x \in F \mid \forall i \in N, \; x_i \; P_i \; y_i\}].$$

Pareto-Efficiency and Partial Pareto have been introduced in Suzumura [26], under the denominations of Exclusion Pareto and Inclusion Pareto, respectively. There are interesting ARSOFs satisfying Partial Pareto but not Pareto-Efficiency. For instance, define

$$S_{\bar{R}}(\mathbf{R}) \equiv \left\{ x \in F \mid \forall i \in N, \; x_i \; R_i \; \frac{\omega}{n} \right\},$$

i.e., $S_{\bar{R}}(\mathbf{R})$ is the set of individually rational allocations from the equal division of resources. This rule satisfies Partial Pareto, with $T = \{(\frac{\omega}{n}, ..., \frac{\omega}{n})\}$.

Let us now consider IIA. The immediate translation is as follows:

IIA (for ARSOF): $\forall \mathbf{R}, \mathbf{R}' \in \mathcal{R}^n$, $\forall x, y \in F$, if $\forall i \in N$,

$$\begin{aligned} x_i \; R_i \; y_i \quad &\Leftrightarrow \quad x_i \; R'_i \; y_i \\ y_i \; R_i \; x_i \quad &\Leftrightarrow \quad y_i \; R'_i \; x_i, \end{aligned}$$

then $[x \in S_{\bar{R}}(\mathbf{R})$ or $y \notin S_{\bar{R}}(\mathbf{R})] \Leftrightarrow [x \in S_{\bar{R}}(\mathbf{R}')$ or $y \notin S_{\bar{R}}(\mathbf{R}')]$.

Interestingly, notice that, since SOFs yield complete orderings, the original IIA axiom can equivalently be written with the conclusion

$$x \; \bar{R}(\mathbf{R}) \; y \Leftrightarrow x \; \bar{R}(\mathbf{R}') \; y$$

or the conclusion

$$x \; \bar{P}(\mathbf{R}) \; y \Leftrightarrow x \; \bar{P}(\mathbf{R}') \; y.$$

As a consequence, the above IIA for ARSOF could equivalently be concluded by

$$[x \in S_{\bar{R}}(\mathbf{R}) \text{ and } y \notin S_{\bar{R}}(\mathbf{R})] \Leftrightarrow [x \in S_{\bar{R}}(\mathbf{R}') \text{ and } y \notin S_{\bar{R}}(\mathbf{R}')],$$

which makes it transparent how demanding it is. It requires that if an allocation is selected while another is not, this does not change when individual preferences relative to these two allocations remain the same, independently of preferences over other allocations.

The translation of Non-Dictatorship is as follows. Let $X \subset F$ be given.

Non-Dictatorship (over X) (for ARSOF): There does not exist $i_0 \in N$ such that:

$$\forall \mathbf{R} \in \mathcal{R}^n, \forall x, y \in X,\ x_{i_0}\ P_{i_0}\ y_{i_0} \Rightarrow x \in S_{\bar{R}}(\mathbf{R}) \text{ and } y \notin S_{\bar{R}}(\mathbf{R}).$$

It is obvious that, as explained in the previous section, this axiom will be trivially satisfied by ARSOFs. For any i_0, one can find x_{i_0}, y_{i_0}, z_{i_0} such that $x_{i_0}\ P_{i_0}\ y_{i_0}\ P_{i_0}\ z_{i_0}$, and if i_0 were a dictator, this would imply $y \in S_{\bar{R}}(\mathbf{R})$ and also $y \notin S_{\bar{R}}(\mathbf{R})$, a contradiction.

More interestingly, the Anonymity axiom then boils down to the following simple standard condition:

Anonymity (for ARSOF): $\forall \pi \in \Pi$, $\forall \mathbf{R} \in \mathcal{R}^n$, $\forall x \in S_{\bar{R}}(\mathbf{R})$, $\pi(x) \in S_{\bar{R}}(\pi(\mathbf{R}))$.

5 IIA and Allocation Rules

It is known from Wilson's theorem (Wilson [34]) that IIA is a very strong axiom. When applied to allocation rules, the fact that IIA is very strong is captured in the following result, which says that it implies that social preferences are totally independent of individual preferences.

Proposition 1 *An ARSOF $\bar{R}$ satisfies IIA if and only if $\bar{R}$ is a constant function.*

The proof is based on a simple argument, which may be summarized as follows. Consider allocation $x^0 \in F$ which gives ω to agent i_0, and 0 to all other agents. Let $x \in F$ be another feasible allocation. Due to the strict monotonicity of preferences, individual preferences over x and x^0 are the same on the whole domain $\mathcal{R}$. Therefore, by IIA, if x is selected while x^0 is not for some $\mathbf{R} \in \mathcal{R}^n$, this must also hold for all $\mathbf{R} \in \mathcal{R}^n$, and similarly if x^0 is selected while x is not.

Proof. It is obvious that a constant ARSOF satisfies IIA. For the converse, choose $i_0 \in N$ and define $x^0 \in F$ by $x^0_{i_0} = \omega$ (and $x^0_i = 0$ for all $i \neq i_0$). If for all $\mathbf{R} \in \mathcal{R}^n$ one has $S_{\bar{R}}(\mathbf{R}) = F$, then $S_{\bar{R}}$ is a constant function. Suppose then that this is not the case, and let $\mathbf{R} \in \mathcal{R}^n$ be such that $S_{\bar{R}}(\mathbf{R}) \neq F$.

First case: $x^0 \in S_{\bar{R}}(\mathbf{R})$. Take any $y \notin S_{\bar{R}}(\mathbf{R})$. By monotonicity of preferences, for all $\mathbf{R}' \in \mathcal{R}^n$,

$$\forall i \in N,\ x^0_i\ R_i\ y_i \Leftrightarrow x^0_i\ R'_i\ y_i \text{ and } y_i\ R_i\ x^0_i \Leftrightarrow y_i\ R'_i\ x^0_i.$$

Therefore $x^0 \in S_{\bar{R}}(\mathbf{R}')$ and $y \notin S_{\bar{R}}(\mathbf{R}')$. The latter implies $F \setminus S_{\bar{R}}(\mathbf{R}) \subset F \setminus S_{\bar{R}}(\mathbf{R}')$. Since $x^0 \in S_{\bar{R}}(\mathbf{R}')$, one can show by a symmetrical argument that $F \setminus S_{\bar{R}}(\mathbf{R}') \subset F \setminus S_{\bar{R}}(\mathbf{R})$ implying $S_{\bar{R}}(\mathbf{R}') = S_{\bar{R}}(\mathbf{R})$.

Second case: $x^0 \notin S_{\bar{R}}(\mathbf{R})$. Take any $x \in S_{\bar{R}}(\mathbf{R})$. By monotonicity of preferences, for all $\mathbf{R}' \in \mathcal{R}^n$,

$$\forall i \in N,\ x_i^0\ R_i\ x_i \Leftrightarrow x_i^0\ R_i'\ x_i \text{ and } x_i\ R_i\ x_i^0 \Leftrightarrow x_i\ R_i'\ x_i^0.$$

Therefore $x^0 \notin S_{\bar{R}}(\mathbf{R}')$ and $x \in S_{\bar{R}}(\mathbf{R}')$. Hence, $S_{\bar{R}}(\mathbf{R}) \subset S_{\bar{R}}(\mathbf{R}')$. Similarly, by a symmetrical argument based on $x^0 \notin S_{\bar{R}}(\mathbf{R}')$, one can show that $S_{\bar{R}}(\mathbf{R}') \subset S_{\bar{R}}(\mathbf{R})$. ■

Contrary to what one might expect, this does not exactly entail an Arrovian impossibility. In fact, there are ARSOFs satisfying IIA and Pareto conditions (and trivially, Non-Dictatorship).

Let us first examine the implication of IIA together with the weakest of our Pareto conditions, namely Partial Pareto. The message of the following proposition is that even with the weakest version of the Pareto conditions, under IIA we are not allowed much room to consider various ARSOFs.

One may get an intuition for the following proposition by considering how an ARSOF $\bar{R}$ may satisfy Pareto-Efficiency (which is stronger than Partial Pareto) and IIA. By the previous result, it must be constant. Now, the only allocations which are Pareto-efficient independently of individual preferences are the allocations like x^0 above, in which one agent receives all of the available resources. With Partial Pareto, a few other possibilities are permitted. Either one selects only allocations in which one agent receives all of the available resources, or one must select all of the allocations in which everyone receives some amount of the resources. Let F^* be the set of feasible allocations with no zero bundle:

$$F^* = \{x \in F \mid \forall i \in N,\ x_i \neq 0\}.$$

Proposition 2 *If an ARSOF $\bar{R}$ satisfies Partial Pareto and IIA, then either for all $\mathbf{R} \in \mathcal{R}^n$,*

$$S_{\bar{R}}(\mathbf{R}) \subseteq \{x \in F \mid \exists i \in N,\ x_i = \omega\}$$

or for all $\mathbf{R} \in \mathcal{R}^n$,

$$F^* \subseteq S_{\bar{R}}(\mathbf{R}).$$

Proof. Let $\mathbf{R} \in \mathcal{R}^n$ be given. Suppose that

$$S_{\bar{R}}(\mathbf{R}) \nsubseteq \{x \in F \mid \exists i \in N,\ x_i = \omega\},$$

that is, there exists $y \in S_{\bar{R}}(\mathbf{R})$ such that for all $i \in N$, $y_i < \omega$. We may assume that $y \neq 0$. For if $y = 0$, then there exists $y' \in F$ such that $y' \gg 0$, and hence for all $j \in N$, $y_j'\ P_j\ y_j$. Since $\bar{R}$ satisfies Partial Pareto, we have $y' \in S_{\bar{R}}(\mathbf{R})$.

Thus, without loss of generality, assume that $0 < y_1 < \omega$. We need to show that $F^* \subseteq S_{\bar{R}}(\mathbf{R})$.

Step 1: We show that int $F \equiv \{x \in F \mid \forall i \in N, x_i \gg 0\} \subseteq S_{\bar{R}}(\mathbf{R})$.

Since $0 < y_1 < \omega$, there are $k, m \in \{1, \ldots, \ell\}, k \neq m$ such that $y_{1k} > 0$ and $y_{1m} < \omega_m$. Without loss of generality, assume that $y_{11} > 0$ and $y_{12} < \omega_2$.

Define $z \in F$ as follows:

(1) $z_{11} = 0$ and $z_{12} = \omega_2$,

(2) for all $i \in N$ with $i \neq 1$, $z_{i1} = y_{i1} + \frac{y_{11}}{n-1}$ and $z_{i2} = 0$, and
(3) for all $j \in N$ and all $k \in \{1, \ldots, \ell\}$ with $k \neq 1, 2$, $z_{ik} = y_{ik}$.

Let $\mathbf{R}^0 = (R_1^0, \ldots, R_n^0)$ be the profile of preferences represented by the following utility functions:

$$\begin{aligned} u_1^0(x_1) &= x_{12} + \frac{1}{r_1} \sum_{m \neq 2} x_{1m}, \\ \forall i &\in N,\ i \neq 1,\ u_i^0(x_i) = x_{i1} + \frac{1}{r_i} \sum_{m \neq 1} x_{im}, \end{aligned}$$

with

$$\begin{aligned} r_1 &> \frac{y_{11}}{\omega_2 - y_{12}} \\ \forall i &\in N,\ i \neq 1,\ r_i > (n-1)\frac{y_{i2}}{y_{11}}. \end{aligned}$$

Then, for all $j \in N$, $z_j\ P_j^0\ y_j$. Since $\bar{R}$ satisfies IIA, from Proposition 1, it is a constant function. Hence, $y \in S_{\bar{R}}(\mathbf{R}^0) = S_{\bar{R}}(\mathbf{R})$. Then, by Partial Pareto, $z \in S_{\bar{R}}(\mathbf{R}^0)$.

To show that int $F \subseteq S_{\bar{R}}(\mathbf{R})$, let $t \in$ int F. Let $\mathbf{R}^1 = (R_1^1, \ldots, R_n^1)$ be the profile of preferences represented by the following utility functions:

$$\begin{aligned} u_1^1(x_1) &= x_{11} + \frac{1}{s_1} \sum_{m \neq 1} x_{1m}, \\ \forall i &\in N,\ i \neq 1,\ u_i^1(x_i) = x_{i2} + \frac{1}{s_i} \sum_{m \neq 2} x_{im}, \end{aligned}$$

with

$$\begin{aligned} s_1 &> \frac{\sum_{m \neq 1} (z_{1m} - t_{1m})}{t_{11}} \\ \forall i &\in N,\ i \neq 1,\ s_i > \frac{\sum_{m \neq 2} (z_{im} - t_{im})}{t_{i2}}. \end{aligned}$$

For all $j \in N$, $t_j\ P_j^1\ z_j$. Because $z \in S_{\bar{R}}(\mathbf{R}^0)$ and $S_{\bar{R}}$ is constant, we have $z \in S_{\bar{R}}(\mathbf{R}^1)$. Then, by Partial Pareto, $t \in S_{\bar{R}}(\mathbf{R}^1)$. Hence, $t \in S_{\bar{R}}(\mathbf{R})$.

Step 2: We show that $F^* \subseteq S_{\bar{R}}(\mathbf{R})$.

Let $y \in F^*$. Then, for all $i \in N$, $y_i \neq 0$. Let $t \in$ int F be chosen so that for each $i \in N$, there is $k(i) \in \{1, \ldots, \ell\}$ such that $0 < t_{ik(i)} < y_{ik(i)}$. Let $\mathbf{R}' = (R_1', \ldots, R_n')$ be the profile of preferences represented by the following utility functions:

$$u_i(x_i) = x_{ik(i)} + \frac{1}{v_i} \sum_{m \neq k(i)} x_{im},$$

with

$$v_i > \frac{\sum_{m \neq k(i)} (t_{im} - y_{im})}{y_{ik(i)} - t_{ik(i)}}.$$

For all $i \in N$, $y_i \; P_i' \; t_i$. Because $t \in S_{\bar{R}}(\mathbf{R})$ and $S_{\bar{R}}$ is constant, we have $t \in S_{\bar{R}}(\mathbf{R}')$. Then, by Partial Pareto, $y \in S_{\bar{R}}(\mathbf{R}')$. Hence, since $S_{\bar{R}}$ is constant, $y \in S_{\bar{R}}(\mathbf{R})$. ■

A direct implication of Proposition 2 is that if one requires essential single-valuedness of an ARSOF $\bar{R}$ in addition to Partial Pareto and IIA, then the associated AR $S_{\bar{R}}$ must be the usual "dictatorial" AR considered in the fair allocation literature, namely the AR which always gives all resources to the same individual. It should then be noted that, even with the weakest version of the Pareto conditions, which does *not* require selected allocations to be Pareto-efficient, IIA and essential single-valuedness together lead us to the version of "dictatorship" in fair allocation theory. Notice that this "dictatorship" is different from the Arrovian dictatorship as defined in our Non-Dictatorship condition. A "dictator" in fair allocation theory can impose his strict preferences only for his top choice in relation to all other alternatives. Again, this is because ARSOFs have at most two indifference classes.

Corollary 1 *An ARSOF $\bar{R}$ satisfies Partial Pareto, IIA and is essentially single-valued if and only if*

$$\exists i \in N, \; \forall \mathbf{R} \in \mathcal{R}^n, \; S_{\bar{R}}(\mathbf{R}) = \{x \in F \mid x_i = \omega\}.$$

If one requires Pareto-Efficiency, which is stronger than Partial Pareto, then, without requiring essential single-valuedness, one can only get ARSOFs that select allocations in which someone gets all resources. However, this does not contradict Anonymity. In fact, together with Anonymity, one gets a full characterization of an ARSOF, as stated in the following theorem. The ARSOF thus characterized is anonymous in the sense that no agent is excluded from the chance to get all resources.

Theorem 1 *If an ARSOF $\bar{R}$ satisfies Pareto-Efficiency and IIA, then*

$$\forall \mathbf{R} \in \mathcal{R}^n, \; S_{\bar{R}}(\mathbf{R}) \subseteq \{x \in F \mid \exists i \in N, \; x_i = \omega\}.$$

An ARSOF $\bar{R}$ satisfies Pareto-Efficiency, IIA and Anonymity if and only if

$$\forall \mathbf{R} \in \mathcal{R}^n, \; S_{\bar{R}}(\mathbf{R}) = \{x \in F \mid \exists i \in N, \; x_i = \omega\}.$$

Proof. By Proposition 2 and Pareto-Efficiency, for all $\mathbf{R} \in \mathcal{R}^n$,

$$S_{\bar{R}}(R) \subseteq \{x \in F \mid \exists i \in N, \; x_i = \omega\}.$$

Since $\bar{R}$ is a constant, for all $\mathbf{R}, \mathbf{R}' \in \mathcal{R}^n$,

$$\{i \in N \mid \exists x \in S_{\bar{R}}(\mathbf{R}), \; x_i = \omega\} = \{i \in N \mid \exists x \in S_{\bar{R}}(\mathbf{R}'), \; x_i = \omega\}.$$

Therefore Anonymity requires

$$\{i \in N \mid \exists x \in S_{\bar{R}}(\mathbf{R}), \; x_i = \omega\} = N.$$

■

Even though the allocation rule characterized in Theorem 1 above is fully anonymous, it is not appealing because it selects only extremely unequal allocations. A minimal requirement of equality is the following:

Equal Treatment of Equals (for ARSOF): $\forall \mathbf{R} \in \mathcal{R}^n$, $\forall x \in S_{\bar{R}}(\mathbf{R})$, $\forall i, j \in N$, if $R_i = R_j$, then $x_i \, I_i \, x_j$.

One may notice that any ARSOF $\bar{R}$ satisfying Anonymity and essential single-valuedness necessarily satisfies Equal Treatment of Equals.

From Theorem 1 we immediately deduce:

Corollary 2 *There is no ARSOF satisfying Pareto-Efficiency, IIA and Equal Treatment of Equals.*[10] *There is no essentially single-valued ARSOF satisfying Pareto-Efficiency, IIA and Anonymity.*

Theorem 1 is interesting not only in its content but also in what it implies about all allocation rules of the fair allocation literature. Since these rules typically satisfy Pareto-Efficiency and Anonymity, and do not give all resources to one individual, they must all violate IIA. Proposition 1 gave the same conclusion even more immediately, since these allocation rules are not constant.

More importantly, the analysis in this section reveals that the possibility results of the theory of fair allocation are not due to the weakening of Weak Pareto into Pareto-Efficiency. Since it is this weakening that allows social preferences to be coarse, this means that the explanation we are looking for does not lie in the fact that allocation rules yield only coarse social preferences. The above results show that the only way for allocation rules to be minimally satisfactory is to violate IIA. *Violation of IIA by ARs is therefore the desired explanation for the contrast between the Arrovian theory of social choice and the theory of fair allocation.*

6 Weakening Independence of Irrelevant Alternatives

At this stage, one may ask in what sense IIA is violated in the theory of fair allocation or, more precisely, what additional information is taken into account by ARSOFs, that is forbidden by IIA.

In the theory of social choice, the main approach with respect to information has been, following Sen [23; 24] in particular, to introduce richer information about utilities. The theory of fair allocation, in contrast, has remained faithful to Arrow's initial project and usually retains only ordinal and interpersonally non-comparable information about preferences. If it introduces more information, it is about preferences, not about utilities. That is, preferences about "irrelevant" alternatives are taken into account by ARs.

[10] A slightly different proof obtains by showing that the only constant ARSOF satisfying Equal Treatment of Equals selects the egalitarian allocation giving ω/n to every agent, which is not Pareto-efficient in general.

It is possible to weaken IIA so as to take account of "irrelevant" alternatives (but not utilities) by strengthening the premise of the axiom in an appropriate way. This attempts brings us into several variants of the axiom, which will be introduced now. In so doing we rely here on previous works by Hansson [15], Fleurbaey and Maniquet [10], and the companion paper Fleurbaey, Suzumura and Tadenuma [13].

A first kind of additional information is contained in the marginal rates of substitution at the allocations to be compared. For efficient allocations, shadow prices enable one to compute the relative implicit income shares of different agents, thereby potentially providing a relevant measure of inequalities in the distribution of resources. Therefore, taking account of marginal rates of substitution is a natural extension of the informational basis of social choice theory in economic environments. Let $C(x_i, R_i)$ denote the cone of price vectors that support the upper contour set for R_i at x_i :

$$C(x_i, R_i) = \{p \in \mathbb{R}^\ell \mid \forall y \in \mathbb{R}^\ell_+,\ py = px_i \Rightarrow x_i R_i y\}.$$

When preferences R_i are strictly monotonic, one has $C(x_i, R_i) \subset \mathbb{R}^\ell_{++}$ whenever $x_i \gg 0$.

One then can require the ranking of two allocations to depend on individual preferences between these two allocations *and* also on marginal rates of substitution at these allocations, but on nothing else:

IIA except Marginal Rates of Substitution (IIA-MRS): $\forall x, y \in F$, $\forall \mathbf{R}, \mathbf{R}' \in \mathcal{R}^n$, if $\forall i \in N$,

$$\begin{array}{rcl} x_i \ R_i \ y_i & \Leftrightarrow & x_i \ R'_i \ y_i \\ y_i \ R_i \ x_i & \Leftrightarrow & y_i \ R'_i \ x_i \\ C(x_i, R_i) & = & C(x_i, R'_i) \\ C(y_i, R_i) & = & C(y_i, R'_i), \end{array}$$

then $x \ \bar{R}(\mathbf{R}) \ y \Leftrightarrow x \ \bar{R}(\mathbf{R}') \ y$.

Marginal rates of substitution give an infinitesimally local piece of information about preferences at given allocations. A further extension of the informational basis allows the SOF to take account of finite parts of indifference hypersurfaces. The *indifference sets* are defined as

$$I(x_i, R_i) = \{z \in \mathbb{R}^\ell_+ \mid z \ I_i \ x_i\}.$$

It is natural to focus on the part of indifference sets which lies within the feasible set. However, when considering any pair of allocations, the two allocations may need different amounts of total resources to be feasible and the global set F need not be relevant in its entirety. Therefore we need to introduce the following notions. The smallest amount of total resources which makes two allocations x and y feasible can be defined by $\omega(x, y) = (\omega_1(x, y), ..., \omega_\ell(x, y))$, where for all $k \in \{1, ..., \ell\}$:

$$\omega_k(x, y) = \max \left\{ \sum_{i \in N} x_{ik}, \sum_{i \in N} y_{ik} \right\}.$$

For each vector $t \in \mathbb{R}^\ell_+$, define the set $\Omega(t) \subset \mathbb{R}^\ell_+$ by

$$\Omega(t) = \left\{ z \in \mathbb{R}^\ell_+ \mid z \leq t \right\}.$$

The following axiom captures the idea that the ranking of two allocations should depend only on the indifference sets, and on preferences over the minimal subset in which the two allocations are feasible.

IIA except Indifference Sets on Feasible Allocations (IIA-ISFA): $\forall x, y \in F$, $\forall \mathbf{R}, \mathbf{R}' \in \mathcal{R}^n$, if $\forall i \in N$,

$$\begin{aligned} I(x_i, R_i) \cap \Omega(\omega(x, y)) &= I(x_i, R_i') \cap \Omega(\omega(x, y)) \\ I(y_i, R_i) \cap \Omega(\omega(x, y)) &= I(y_i, R_i') \cap \Omega(\omega(x, y)), \end{aligned}$$

then $x \; \bar{R}(\mathbf{R}) \; y \Leftrightarrow x \; \bar{R}(\mathbf{R}') \; y$.

It is immediate from the definitions that

$$\begin{array}{ccc} \text{IIA} & \Rightarrow & \text{IIA-MRS} \\ \Downarrow & & \\ \text{IIA-ISFA} & & \end{array}$$

Notice that IIA-MRS does not imply IIA-ISFA because the set $I(x_i, R_i) \cap \Omega(\omega(x, y))$ does not always provide enough information to determine $C(x_i, R_i)$.[11]

It is also worthwhile here introducing a couple of independence conditions for ARs, which are closely related to IIA and its variants. Such conditions are quite common in the fair allocation literature. We will formulate them here for ARSOFs.

The first one, dealing with marginal rates of substitution, is essentially Nagahisa's [20] 'Local Independence':[12]

Independence of Preferences except MRS (IP-MRS): $\forall x \in F$, $\forall \mathbf{R}, \mathbf{R}' \in \mathcal{R}^n$, if $\forall i \in N$,

$$C(x_i, R_i) = C(x_i, R_i'),$$

then $x \in S_{\bar{R}}(\mathbf{R}) \Leftrightarrow x \in S_{\bar{R}}(\mathbf{R}')$.

The next axiom says that only the parts of indifference sets concerning feasible allocations should matter.

Independence of Preferences except Indifference Sets on Feasible Allocations (IP-ISFA): $\forall x \in F$, $\forall \mathbf{R}, \mathbf{R}' \in \mathcal{R}^n$, if $\forall i \in N$,

$$I(x_i, R_i) \cap \Omega(\omega) = I(x_i, R_i') \cap \Omega(\omega),$$

then $x \in S_{\bar{R}}(\mathbf{R}) \Leftrightarrow x \in S_{\bar{R}}(\mathbf{R}')$.

Although these independence conditions may seem restrictive, they are actually not really stronger than the previous IIA axioms.

[11] It does, however, when every good is consumed by at least two agents in x.

[12] See also Yoshihara [35].

Proposition 3 *On the class of ARSOFs that never select the null allocation* $0 = (0, \ldots, 0)$*, IIA-MRS* $\Rightarrow$ *IP-MRS, and IIA-ISFA* $\Rightarrow$ *IP-ISFA.*

Proof. IIA-MRS $\Rightarrow$ IP-MRS. Let $x \in S_{\bar{R}}(\mathbf{R})$ and $\mathbf{R}'$ be such that for all $i \in N$, $C(x_i, R_i') = C(x_i, R_i)$. Notice that $0 = (0, \ldots, 0) \notin S_{\bar{R}}(\mathbf{R})$. Since for all $i \in N$, $C(0, R_i') = C(0, R_i) = \mathbb{R}^{\ell}_{+}$, and $x_i \; R_i \; 0 \Leftrightarrow x_i \; R_i' \; 0$, and $0 \; R_i \; x_i \Leftrightarrow 0 \; R_i' \; x_i$, it follows from IIA-MRS that $x \in S_{\bar{R}}(\mathbf{R}')$ and $0 \notin S_{\bar{R}}(\mathbf{R}')$.

IIA-ISFA$\Rightarrow$IP-ISFA. Let $x \in S_{\bar{R}}(\mathbf{R})$ and $\mathbf{R}'$ be such that for all $i \in N$, $I(x_i, R_i) \cap \Omega(\omega) = I(x_i, R_i') \cap \Omega(\omega)$. Notice that for all $i \in N$, $I(0, R_i') = I(0, R_i) = \{0\}$. Then, by IIA-ISFA, $x \in S_{\bar{R}}(\mathbf{R}')$. ■

It is also easy to check that IP-MRS implies IIA-MRS, and that, for ARSOFs which never select allocations x such that $\sum_{i \in N} x_i \neq \omega$, IP-ISFA implies IIA-ISFA. In other words, for all practical purposes, the distinction between the IP axioms introduced here and their IIA counterparts is negligible.

The question we may now consider is how much IIA needs to be weakened, or how much additional information is needed in order to obtain the existence of a satisfactory AR (or ARSOF).

Our first result is that with IIA-MRS, a possibility is obtained, but there remains a difficulty about essential single-valuedness. As can be expected from the examples in Section 3, IIA-MRS is satisfied by the Egalitarian Walrasian ARSOF $\bar{R}_{S_W}$, along with many good properties. But, as it is well known, the Egalitarian Walrasian AR is not essentially single-valued. Now, in the proof below we find a subdomain on which any ARSOF satisfying Pareto-Efficiency, IIA-MRS and Equal Treatment of Equals coincides exactly with $\bar{R}_{S_W}$ on this subdomain, even though $\bar{R}_{S_W}$ is still not essentially single-valued on this subdomain.

Theorem 2 *There exists an ARSOF satisfying Pareto-Efficiency, IIA-MRS, Equal Treatment of Equals and Anonymity. There is no essentially single-valued ARSOF satisfying Pareto-Efficiency, IIA-MRS and Equal Treatment of Equals. There is no essentially single-valued ARSOF satisfying Pareto-Efficiency, IIA-MRS and Anonymity.*

Proof. The possibility is illustrated by the Egalitarian Walrasian ARSOF $\bar{R}_{S_W}$.

The second impossibility is implied by the first impossibility because essential single-valuedness and Anonymity imply Equal Treatment of Equals. To show the first impossibility, suppose, to the contrary, that there exists an essentially single-valued ARSOF $\bar{R}$ satisfying Pareto-Efficiency, IIA-MRS and Equal Treatment of Equals. By Pareto-Efficiency, for all $\mathbf{R} \in \mathcal{R}^n$, $0 = (0, \ldots, 0) \notin S_{\bar{R}}(\mathbf{R})$. Hence, from Proposition 3, $\bar{R}$ satisfies IP-MRS.

Let $\mathcal{R}^*$ be the subset of $\mathcal{R}$ such that any $R \in \mathcal{R}^*$ is representable by a utility function of the following kind:

$$u(x_1, ..., x_\ell) = f_1(x_1) + ... + f_\ell(x_\ell),$$

where for all $k \in \{1, ..., \ell\}$, f_k is continuous, increasing, concave, and differentiable over $\mathbb{R}_{++}$, with $\lim_{x \to 0} f_k'(x) = +\infty$. The relevant property of this domain is that for all

$\mathbf{R} \in (\mathcal{R}^*)^n$,

$$E(\mathbf{R}) \subseteq \{x \in \mathbb{R}_+^\ell \mid \forall i \in N,\ x_i \gg 0 \text{ or } x_i = 0\}.$$

Let $\mathbf{R} \in (\mathcal{R}^*)^n$ be given.

Firstly, suppose that there is $x \in S_{\bar{R}}(\mathbf{R}) \setminus S_W(\mathbf{R})$. By Pareto-Efficiency $x \in E(\mathbf{R})$. Hence, we have $x_i \gg 0$ or $x_i = 0$ for all $i \in N$, and by differentiability of preferences there is a shadow price vector $p \in \mathbb{R}_{++}^\ell$ such that

$$\forall i \in N,\ C(x_i, R_i) = \{\lambda p \mid \lambda \in \mathbb{R}_{++}\} \text{ or } x_i = 0.$$

For this p, define $R^p \in \mathcal{R}$ by

$$\forall z, z' \in \mathbb{R}_+^\ell,\ z\ R^p\ z' \Leftrightarrow p \cdot z \geq p \cdot z'.$$

Let $\mathbf{R}^p = (R^p, ..., R^p) \in \mathcal{R}^n$. By IP-MRS, $x \in S_{\bar{R}}(\mathbf{R}^p)$. Since $x \notin S_W(\mathbf{R})$, there exist $i, j \in N$ such that $x_i\ P^p\ x_j$, in contradiction to Equal Treatment of Equals. As a consequence, $S_{\bar{R}}(\mathbf{R}) \subset S_W(\mathbf{R})$.

Secondly, suppose that there is $x \in S_W(\mathbf{R}) \setminus S_{\bar{R}}(\mathbf{R})$. For all $i \in N$, let $\mathbf{R}' \in (\mathcal{R}^*)^n$ be a profile of homothetic (a given R in $\mathcal{R}^*$ is homothetic if all its component functions f_k are homogeneous of the same degree) and strictly convex preferences satisfying

$$\forall i \in N,\ C(x_i, R_i') = C(x_i, R_i).$$

We have $x \in S_W(\mathbf{R}')$. Moreover, by Theorem 1 in Eisenberg [7], all allocations in $S_W(\mathbf{R}')$ are Pareto-indifferent. By strict convexity of preferences, one therefore has $S_W(\mathbf{R}') = \{x\}$. Since, by the previous argument, $S_{\bar{R}}(\mathbf{R}') \subset S_W(\mathbf{R}')$, we have $S_{\bar{R}}(\mathbf{R}') = \{x\}$. By IP-MRS, $x \in S_{\bar{R}}(\mathbf{R})$, which is a contradiction. Therefore $S_W(\mathbf{R}) \subset S_{\bar{R}}(\mathbf{R})$.

In conclusion, $S_{\bar{R}}(\mathbf{R}) = S_W(\mathbf{R})$ for all $\mathbf{R} \in (\mathcal{R}^*)^n$. But S_W is not essentially single-valued on the whole domain $(\mathcal{R}^*)^n$. This contradicts essential single-valuedness of $S_{\bar{R}}$. ■

Only with IIA-ISFA do we really obtain a full possibility result, with the Pazner-Schmeidler ARSOF $\bar{R}_{S_{PS}}$.

Theorem 3 *There exists an essentially single-valued ARSOF satisfying Pareto-Efficiency, IIA-ISFA, Anonymity and Equal Treatment of Equals.*

Proof. Consider the Pazner-Schmeidler ARSOF $\bar{R}_{S_{PS}}$, defined at the end of subsection 4.1. It obviously satisfies Pareto-Efficiency, Anonymity and Equal Treatment of Equals. To check that it satisfies IIA-ISFA, let $x, y \in F$ and $\mathbf{R}, \mathbf{R}' \in \mathcal{R}^n$ be such that for all $i \in N$,

$$\begin{aligned} I(x_i, R_i) \cap \Omega(\omega(x, y)) &= I(x_i, R_i') \cap \Omega(\omega(x, y)) \\ I(y_i, R_i) \cap \Omega(\omega(x, y)) &= I(y_i, R_i') \cap \Omega(\omega(x, y)), \end{aligned}$$

and $x \in S_{PS}(\mathbf{R})$ and $y \notin S_{PS}(\mathbf{R})$. Let $\alpha \in \mathbb{R}_+$ be such that for all $i \in N$, $x_i\ I_i\ \alpha\omega$. Then, necessarily $\alpha < 1$. Notice that $\sum_{i \in N} x_i = \omega$ because $x \in E(\mathbf{R})$. Hence, $\Omega(\omega(x, y)) = \Omega(\omega)$, and $\alpha\omega \in \Omega(\omega(x, y))$. Together with the above equalities, we deduce that $x \in S_{PS}(\mathbf{R}')$ and $y \notin S_{PS}(\mathbf{R}')$. ■

7 Under Weak Pareto, Social Ordering Functions Need More Information

From the previous results, we now know that violation of IIA is crucial for the possibility results of fair allocation, and that introducing additional information about marginal rates of substitution is almost sufficient, while information about indifference surfaces on feasible allocations is fully sufficient. Such results are obtained for ARSOFs, that is, under the condition that social preferences are coarse and satisfy only Pareto-Efficiency. The question we now want to examine is whether SOFs satisfying the full Weak Pareto condition, and therefore corresponding to fine-grained social preferences, are possible with the same additional information, or whether they need more information, i.e. further weakenings of IIA. In other words, is there a trade-off between Pareto conditions and independence conditions?

Fleurbaey and Maniquet [10], in this model, showed that there exist many SOFs satisfying Weak Pareto, Anonymity and the following weak version of IIA:

IIA except Whole Indifference Sets (IIA-WIS): $\forall x, y \in F$, $\forall \mathbf{R}, \mathbf{R}' \in \mathcal{R}^n$, if $\forall i \in N$,

$$\begin{aligned} I(x_i, R_i) &= I(x_i, R'_i) \\ I(y_i, R_i) &= I(y_i, R'_i), \end{aligned}$$

then $x \; \bar{R}(\mathbf{R}) \; y \Leftrightarrow x \; \bar{R}(\mathbf{R}') \; y$.

This axiom is weaker than all IIA axioms considered above, and one may ask what is the minimal amount of information needed by a SOF in order to satisfy Weak Pareto and Anonymity (or Non-Dictatorship). In Fleurbaey, Suzumura and Tadenuma [13], we showed that no SOF $\bar{R}$ satisfies Weak Pareto, Non-Dictatorship (over the subset X of allocations in which no agent has a zero bundle), and either IIA-MRS or IIA-ISFA.

But these results were obtained in the particular case of unbounded resources $F = \mathbb{R}_+^{n\ell}$. The bounded case on which we focus here has attracted less attention in the social choice literature,[13] and here we have the following result.

Theorem 4 *There is no SOF $\bar{R}$ satisfying Weak Pareto, IIA-MRS and Anonymity. There is no SOF $\bar{R}$ satisfying Weak Pareto, IIA-ISFA and Anonymity.*

Proof. For simplicity of exposition, we assume that $\omega > (20, 20, 0, ..., 0)$. When this does not hold, a suitable renormalization of goods allows the rest of the proof to work out. For each $a \in \mathbb{R}_+^{\ell}$, define

$$B(a) = \{b \in \mathbb{R}_+^{\ell} \mid \max_{k \in \{1,\ldots,\ell\}} |b_k - a_k| \leq \frac{1}{10}\}$$

In order to prove the impossibilities, it is convenient to consider different possible sizes of the population.

[13] Exceptions are Bordes, Campbell and Le Breton [4], already quoted, and also Bone [3].

Case $n = 2$. Consider the bundles $x = (8, 1/2, 0, ...)$, $y = (12, 1/2, 0, ...)$, $z = (1/2, 12, 0, ...)$, $w = (1/2, 8, 0, ...)$. Let preferences R_1 and R_2 be defined as follows. On the subset

$$S_1 = \{v \in \mathbb{R}^\ell_+ \mid \forall i \in \{3, ..., \ell\}, v_i = 0 \text{ and } v_2 \leq \min\{v_1, 1\}\}$$

one has

$$v\ R_1\ v' \Leftrightarrow v_1 + 2v_2 \geq v'_1 + 2v'_2,$$

and on the subset

$$S_2 = \{v \in \mathbb{R}^\ell_+ \mid \forall i \in \{3, ..., \ell\}, v_i = 0 \text{ and } v_1 \leq \min\{v_2, 1\}\},$$

one has

$$v\ R_1\ v' \Leftrightarrow 2v_1 + v_2 \geq 2v'_1 + v'_2.$$

On $B(x) \cup B(y)$, one has

$$v\ R_1\ v' \Leftrightarrow v_1 + 2v_2 + \sum_{k=3}^{\ell} v_k \geq v'_1 + 2v'_2 + \sum_{k=3}^{\ell} v'_k,$$

and on $B(z) \cup B(w)$,

$$v\ R_1\ v' \Leftrightarrow 2v_1 + v_2 + \sum_{k=3}^{\ell} v_k \geq 2v'_1 + v'_2 + \sum_{k=3}^{\ell} v'_k.$$

Since

$$w_1 + (1 - w_1) + 2\left[w_2 - 2(1 - w_1)\right] > x_1 + 2x_2$$

and

$$2\left[y_1 - 2(1 - y_2)\right] + y_2 + (1 - y_2) > 2z_1 + z_2,$$

it is possible to complete the definition of R_1 such that $w\ P_1\ x$ and $y\ P_1\ z$. Then define R_2 so that it coincides with R_1 on $S_1 \cup S_2$, and on $B(a)$ for all $a \in \{x, y, z, w\}$. Similarly, it is possible to complete the definition of R_2 such that $x\ P_2\ w$ and $z\ P_2\ y$. Figure 4 illustrates this construction.

If the profile of preferences is $\mathbf{R} = (R_1, R_2)$, by Weak Pareto one has:

$$(y, x)\ \bar{P}(\mathbf{R})\ (z, w) \text{ and } (w, z)\ \bar{P}(\mathbf{R})\ (x, y).$$

If the profile of preferences is $\mathbf{R}' = (R_1, R_1)$, by Anonymity one has:

$$(y, x)\ \bar{I}(\mathbf{R}')\ (x, y) \text{ and } (w, z)\ \bar{I}(\mathbf{R}')\ (z, w).$$

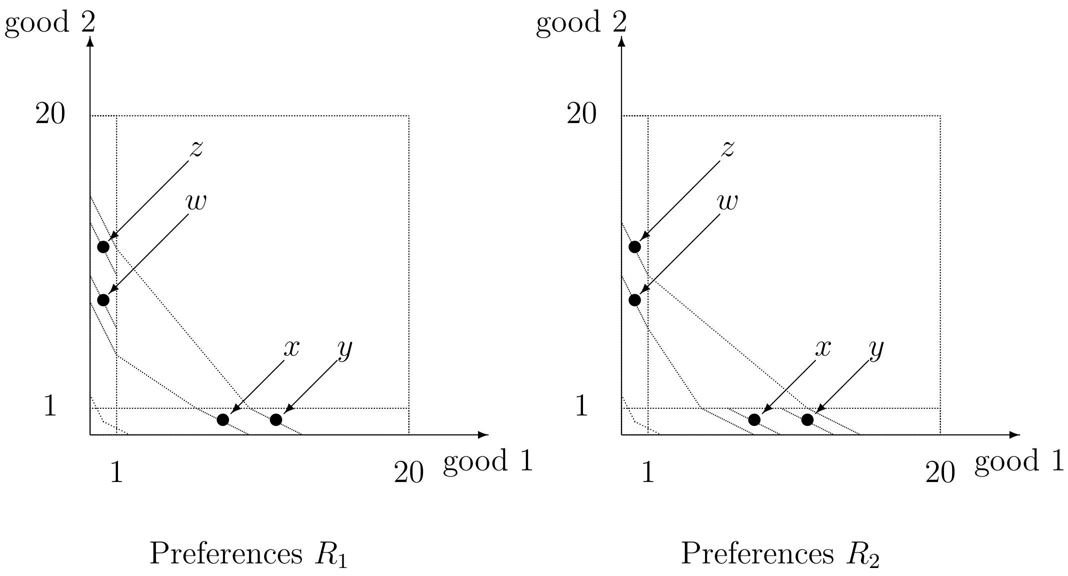

Figure 4: Construction of R_1 and R_2

Since R_1 and R_2 coincide on $S_1 \cup S_2$, and on $B(a)$ for all $a \in \{x, y, z, w\}$, by IIA-MRS or IIA-ISFA, one has:

$$\begin{aligned} (y, x)\ \bar{I}(\mathbf{R}')\ (x, y) &\Leftrightarrow (y, x)\ \bar{I}(\mathbf{R})\ (x, y) \\ \text{and } (w, z)\ \bar{I}(\mathbf{R}')\ (z, w) &\Leftrightarrow (w, z)\ \bar{I}(\mathbf{R})\ (z, w). \end{aligned}$$

By transitivity, one gets $(x, y)\ \bar{P}(\mathbf{R})\ (x, y)$, which is impossible.
Case $n = 3$. Consider the bundles $x = (8, 1/3, 0, ...)$, $y = (12, 1/3, 0, ...)$, $t = (10, 1/3, 0, ...)$, $z = (1/3, 12, 0, ...)$, $w = (1/3, 8, 0, ...)$, $r = (1/3, 10, 0, ...)$. Let preferences R_1, R_2 and R_3 be defined as above on the subset $S_1 \cup S_2$, and on $B(a)$ for all $a \in \{x, y, z, w\}$. Complete their definition so that $y\ P_1\ z$, $w\ P_1\ x$, $t\ P_2\ r$, $z\ P_2\ y$, $x\ P_3\ w$, and $r\ P_3\ t$.

If the profile of preferences is $\mathbf{R} = (R_1, R_2, R_3)$, by Weak Pareto one has:

$$(y, t, x)\ \bar{P}(\mathbf{R})\ (z, r, w) \text{ and } (w, z, r)\ \bar{P}(\mathbf{R})\ (x, y, t).$$

If the profile of preferences is $\mathbf{R}' = (R_1, R_1, R_1)$, by Anonymity one has:

$$(y, t, x)\ \bar{I}(\mathbf{R}')\ (x, y, t) \text{ and } (w, z, r)\ \bar{I}(\mathbf{R}')\ (z, r, w).$$

Since R_1, R_2 and R_3 coincide on $S_1 \cup S_2$, and on $B(a)$ for all $a \in \{x, y, z, w\}$, by IIA-MRS or IIA-ISFA, one has:

$$\begin{aligned} (y, t, x)\ \bar{I}(\mathbf{R}')\ (x, y, t) &\Leftrightarrow (y, t, x)\ \bar{I}(\mathbf{R})\ (x, y, t) \\ \text{and } (w, z, r)\ \bar{I}(\mathbf{R}')\ (z, r, w) &\Leftrightarrow (w, z, r)\ \bar{I}(\mathbf{R})\ (z, r, w). \end{aligned}$$

By transitivity, one gets $(x, y, t)\ \bar{P}(\mathbf{R})\ (x, y, t)$, which is impossible.

Case $n = 2k$. Partition the population into k pairs, and construct an argument similar to the case $n = 2$, with the bundles $x = (8, 1/n, 0, ...)$, $y = (12, 1/n, 0, ...)$, $z = (1/n, 12, 0, ...)$, $w = (1/n, 8, 0, ...)$, and the allocations $(y, x, y, x, ...)$, $(x, y, x, y, ...)$, $(z, w, z, w, ...)$ and $(w, z, w, z, ...)$.
Case $n = 2k + 1$. Partition the population into $k - 1$ pairs and one triple, and construct an argument combining the cases $n = 2$ and $n = 3$, with the bundles $x = (8, 1/n, 0, ...)$, $y = (12, 1/n, 0, ...)$, $t = (10, 1/n, 0, ...)$, $z = (1/n, 12, 0, ...)$, $w = (1/n, 8, 0, ...)$, $r = (1/n, 10, 0, ...)$, and the allocations $(y, x, y, x, ..., y, t, x)$, $(x, y, x, y, ..., x, y, t)$, $(z, w, z, w, ...z, r, w)$ and $(w, z, w, z, ..., w, z, r)$. ■

This result proves that under Weak Pareto, more information about preferences is needed than under Pareto-Efficiency. In that sense, it is true that *the theory of fair allocation, with its coarse orderings, is less demanding in information than the theory of social choice.*

As explained in Fleurbaey, Suzumura and Tadenuma [13], however, one should not conclude from this analysis that full knowledge of indifference curves is needed under Weak Pareto. Define the Pazner-Schmeidler SOF $\bar{R}_{PS}$ as follows: $x\ \bar{R}(\mathbf{R})\ y$ if and only if

$$\min\ \{\alpha \in \mathbb{R}_+ \mid \exists i \in N,\ \alpha\omega\ R_i\ x_i\} \geq \min\ \{\alpha \in \mathbb{R}_+ \mid \exists i \in N,\ \alpha\omega\ R_i\ y_i\}.$$

This SOF satisfies Weak Pareto and Anonymity, even though it only requires knowledge of the intersection of indifference curves with a ray from the origin. In addition, although this SOF does not satisfy IIA-ISFA in the current framework, it can be shown to satisfy IIA-ISFA when only allocations of the subset

$$\left\{ x \in \mathbb{R}_+^{n\ell} \mid \sum_{i \in N} x_i = \omega \right\}$$

with no free disposal, instead of F, are ranked.

8 Toward a Unified Theory

There have been many attempts to import fairness concepts into social choice, and thereby build a unified theory, such as Feldman and Kirman [8], Varian [32], Suzumura [27; 28; 29] and Tadenuma [30]. But they did not focus on the informational requirements to obtain positive results.

Our approach provides a unified framework which covers the theory of social choice and the theory of fair allocation. Because ARs in the theory of fair allocation are isomorphic to ARSOFs in the theory of social choice, and ARSOFs are just a particular kind of SOF, the concept of SOF is comprehensive enough to encompass all relevant notions. This shows how the theory of fair allocation is, rigorously, a part of the theory of social choice.

As a consequence, the way in which possibility results are obtained with ARs, by broadening the informational basis, can be adopted for SOFs, albeit, as shown above, the amount of additional information needed is greater under Weak Pareto. From this perspective, there is no longer any reason to view the theory of social choice as plagued with impossibilities, and no longer any reason for social choice theorists to envy fairness theorists and their positive results. The same recipe for success can be adopted by social choice theorists.[14]

In this section we examine two possible objections to this proposed integration of fair allocation theory into social choice theory. The first objection would go by recollecting that the celebrated Arrow Program of social choice theory consists of two separate steps, viz., (a) the construction of a social preference ordering corresponding to each profile of individual preference orderings; and (b) the construction of a social choice function by means of optimization of social preferences within each set of feasible social alternatives. The first step, which may be called the *preference aggregation stage*, is to determine the *uniform* social objective before the set of feasible social alternatives is revealed. The second step, which may be called the *rational choice stage*, is to determine the rational social choice after the set of feasible social alternatives is revealed. Even though we may construct a coarse social ordering in terms of the fair allocations versus unfair allocations, such an ordering hinges squarely on the specification of the set of feasible allocations. In other words, no social preference ordering, which can be applied *uniformly* to all feasible sets of alternatives, can thereby be generated. Thus, the objection would go, in view of the basic scenario of the Arrow Program of social choice theory, the theory of fair allocation does not really offer much to the preference aggregation stage of social choice theory.

Our response to this objection is that what is called "social choice theory" in this paper actually encompasses the preference aggregation stage of the Arrow program, as presented above, as a special case. We believe that it is quite convenient to see the common formal structure in all exercises of construction of a preference ordering over a set of alternatives, whether this set is determined by feasibility constraints or not. In this paper, the need to compare the social choice approach and the fair allocation approach has led us to retain

$$F = \{x \in \mathbb{R}^{\ell}_{+} \mid x_1 + ... + x_n \leq \omega\}$$

as the relevant set of alternatives. An orthodox vision of the Arrow Program of social choice theory might possibly require the construction of the social preference ordering to be made on the full set $\mathbb{R}^{n\ell}_{+}$, rather than F, but we do not think that the construction of a social preference ordering over F should be excluded from social choice theory for that reason.[15] Moreover, the notion of feasibility itself is multi-faceted. Although F is determined by some feasibility constraints, the set of actually feasible alternatives, in practical applications, is likely to be a strict subset of F. For instance, the political system

[14] For characterizations of SOFs based on fairness axioms, see e.g. Fleurbaey and Maniquet [11; 12].

[15] Arrow [2] himself was actually vague about the set of alternatives in his monograph on social choice. For instance, in the economic example he introduces in chapter 6, section 4, he simply states: 'Suppose that among the *possible* alternatives there are three, none of which gives any individual at least as much of both commodities as any other' (Arrow [2, p.68]; emphasis added). Bordes, Campbell and Le Breton [4] study Arrow's theorem on F as a relevant social choice exercise.

may give special value to a status quo x_0, and restrict attention to another particular alternative x, introduced as a proposed reform of the status quo. In order to decide whether x is better than x_0 or not, a fine-grained ranking of all members of F is quite useful, and a ranking of all members of $\mathbb{R}_+^{n\ell}$ would be perfectly adequate as well, but would be more than needed.

The second objection to our unification would rely on an alternative way of unifying the two theories, which has been elegantly formulated in Fishburn [9] and adapted to economic environments by Le Breton [18]. It consists in broadening the concept of AR, as done in the theory of social choice based on social decision rules (SDR).

Let $\mathcal{F}$ denote the set of non-empty subsets of F, and let $\mathcal{A} \subset \mathcal{F}$. A *social decision rule* (SDR) is a mapping $\bar{S}$ from $\mathcal{R}^n \times \mathcal{A}$ to $\mathcal{F}$ such that for all $\mathbf{R} \in \mathcal{R}^n$, all $A \in \mathcal{A}$, $\bar{S}(\mathbf{R}, A) \subset A$ and $\bar{S}(\mathbf{R}, A) \neq \emptyset$. Each set A is called an agenda, and $\mathcal{A}$ is the class of agendas.

In this approach, an AR is just a particular kind of SDR for which $\mathcal{A} = \{F\}$. By contrast, if $\mathcal{A}$ contains all pairs of allocations $\{x, y\} \subset F$, one can recover an SOF from an SDR whenever the SDR satisfies a choice consistency condition. The derived SOF $\bar{R}_{\bar{S}}$ is then defined by:

$$x \; \bar{R}_{\bar{S}}(\mathbf{R}) \; y \Leftrightarrow x \in \bar{S}(\mathbf{R}, \{x, y\}).$$

In this perspective, the specificity of the theory of fair allocation is that it has a very restricted class of agendas. This expresses the fact that the theory of fair allocation only seeks the good allocations among all feasible ones, whereas the theory of social choice wants to make fine-grained selections in most conceivable agendas.

The fact that possibility results are obtained in the theory of fair allocation is likely to be interpreted, in this approach, as due to the restricted agendas, and this reinforces the usual explanation which opposes fine-grained social preferences and selection. But this would be a hasty conclusion. Arrow's independence condition, applied to SDRs, is formulated as follows in Le Breton [18]:

Independence of Infeasible Alternatives (IIF): $\forall \mathbf{R}, \mathbf{R}' \in \mathcal{R}^n$, $\forall A \in \mathcal{A}$, if $\forall i \in N$,

$$\forall x, y \in A: \; x_i \; R_i \; y_i \Leftrightarrow x_i \; R_i' \; y_i,$$

then $\bar{S}(\mathbf{R}, A) = \bar{S}(\mathbf{R}', A)$.

When the class of agendas is restricted, the amount of information about preferences that may be used by $\bar{S}$ when considering to choose x as against y increases automatically, because the subset A on which preference information is retained becomes larger. Therefore going to restricted agendas has two consequences. First, it makes one go from fine-grained preferences to coarse preferences, as emphasized by the usual explanation of the possibility results in fair allocation theory. Second, and we believe more importantly, it increases the amount of relevant information about preferences, as delineated by IIF. And in the limit where $\mathcal{A} = \{F\}$, the amount of relevant information is maximal. Hence, in this approach, all ARs are indistinguishable in terms of informational basis since all have the same maximal basis.

It is true that in order to identify all the "good" allocations among all feasible allocations, we need anyway to know information about preferences of individuals over the full set F. However, often in reality, we are faced with a different question: Given two allocations x and y, can we say whether one of the two allocations is good while the other is not? In fact, ARs can answer this type of questions. For instance, if x is an egalitarian Walrasian allocation and y is not, one can say that x is good while y is not. As shown in previous sections, information about preferences necessary to derive such an evaluation is much different among various ARs. For example, the Egalitarian Walrasian AR only needs knowledge of marginal rates of substitution at the given two allocations whereas the Pazner-Schmeidler AR needs more global knowledge about indifference surfaces. Such distinctions can be captured by the IIA-MRS axiom, the IIA-ISFA axiom, and other axioms in the SOF approach developed in this chapter. We are thus inclined to think that the SOF approach is more suitable to the analysis of the informational basis of the various theories.[16]

9 Conclusion

Let us briefly summarize the conclusions of this chapter.

1. The allocation rules from the theory of fair allocation do provide social ordering functions, so that Arrow's impossibility theorem of social choice does apply to them.

2. No satisfactory allocation rule satisfies Independence of Irrelevant Alternatives, so that violation of this axiom is the explanation for the possibility results in fair allocation theory, as compared to impossibility results in social choice theory.

3. In the process of weakening the Independence axiom, the introduction of additional information about marginal rates of substitution or about indifference surfaces on possible redistributions of the contemplated allocations is sufficient for satisfactory allocation rules to be obtained.

4. Requiring Weak Pareto, not just Pareto-Efficiency, which implies that social preferences must be more fine-grained than allocation rules, makes it necessary to introduce more information about preferences.

More precisely, the results in this chapter may be summarized as follows:

[16] The literature does, however, contain some examples of finer informational axioms in the SDR setting. In an abstract model with a fixed agenda, Denicolò [5; 6] introduces a pairwise independence axiom on SDRs in order to obtain impossibility results of the Arrovian sort. This axiom says that if two profiles coincide on $\{x, y\}$, and x is selected and y is not under one profile, then y is still not selected under the other profile.

	Weak Pareto	**Pareto Efficiency**
IIA	Arrovian Dictatorship (Arrow's Theorem)	ARSOFs giving someone all resources (Theorem 1)
IIA-MRS	Violation of Anonymity (Theorem 4)	Anonymous ARSOFs with Equal Treatment of Equals (Theorem 2)
IIA-ISFA	Violation of Anonymity (Theorem 4)	Essentially single-valued, anonymous ARSOFs with Equal Treatment of Equals (Theorem 3)
IIA-WIS	Anonymous SOFs (Pazner-Schmeidler [22])	Essentially single-valued, anonymous ARSOFs with Equal Treatment of Equals (Theorem 3)

Table 1: Summary of the Results

As shown clearly by Table 1, weakening Weak Pareto into Pareto Efficiency, and therefore allowing coarse orderings, alone does not make room for satisfactory ARs. Weakening IIA, and thus expanding informational basis for social evaluation of allocations, is essential for positive results in fair allocation theory.

However, as is also clear in the table, whether we seek fully Paretian social orderings or only coarse orderings satisfying Pareto Efficiency does make a difference in *how much expansion* of informational basis is indeed necessary beyond what Arrow's original IIA allows.

We hope that this chapter, more broadly, contributes to clarifying the informational foundations in the theory of social choice and in the theory of fair allocation, and also to clarifying the links and differences between these two theories. Our proposal for a unified theory of social choice, where possibility results from the fairness part can be extended to SOFs, should shake off the negative fame of social choice theory.

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Chapter 7
Ordering Infinite Utility Streams*

1 Introduction

Treating generations equally is one of the basic principles in the utilitarian tradition of moral philosophy. As Sidgwick [20, p.414] observes, "the time at which a man exists cannot affect the value of his happiness from a universal point of view; and [. . .] the interests of posterity must concern a utilitarian as much as those of his contemporaries." This view, which is formally expressed by the anonymity condition, is also strongly endorsed by Ramsey [16].

Following Koopmans [14], Diamond [9] establishes that anonymity is incompatible with the strong Pareto principle when ordering *infinite* utility streams. Moreover, he shows that if anonymity is weakened to *finite* anonymity — which restricts the application of the standard anonymity requirement to situations where utility streams differ in at most a finite number of components — and a continuity requirement is added, an impossibility results again. Hara, Shinotsuka, Suzumura and Xu [12] adapt the well-known strict transfer principle due to Pigou [15] and Dalton [7] to the infinite-horizon context. They show that this principle is incompatible with strong Pareto and continuity even if the social preference is merely required to be acyclical. Basu and Mitra [5] show that strong Pareto, finite anonymity and representability by a real-valued function are incompatible.

Faced with these impossibilities, it seems to us that the most natural assumption to drop is that of continuity or representability. We view the strong Pareto principle and finite anonymity as being on much more solid ground than axioms such as continuity or representability, especially in the context of the ranking of infinite utility streams where these conditions may be considered to be overly demanding. Svensson [22] proves that

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strong Pareto and finite anonymity are compatible by showing that any ordering extension of an infinite-horizon variant of Suppes' [21] grading principle satisfies the required axioms. The Suppes grading principle is a quasi-ordering that combines the Pareto quasi-ordering and finite anonymity. Given Arrow's [1] version of Szpilrajn's [23] extension theorem, this establishes the compatibility result. As noted by Asheim, Buchholz and Tungodden [2], Svensson's possibility result is easily converted into a characterization: ordering extensions of the Suppes grading principles are *the only* orderings satisfying strong Pareto and finite anonymity.

Once the possibility of satisfying these two fundamental axioms is established, another natural question to ask is what orderings satisfy additional desirable properties. Asheim and Tungodden [3] provide a characterization of an infinite-horizon version of the leximin principle by adding an equity-preference condition (the infinite-horizon equivalent of Hammond equity; see Hammond [10]) and a preference-continuity property to strong Pareto and finite anonymity. An infinite-horizon version of utilitarianism is characterized by Basu and Mitra [6] by adding an information-invariance condition to the two fundamental axioms. Furthermore, they narrow down the class of infinite-horizon utilitarian orderings to those resulting from the overtaking criterion (von Weizsäcker [24]). This is accomplished by using a consistency condition in addition to the three axioms characterizing their utilitarian orderings.

In this chapter, we focus on equity properties. One of the most fundamental equity properties (if not *the* most fundamental) is the Pigou-Dalton transfer principle, adapted to the infinite-horizon framework by Hara, Shinotsuka, Suzumura and Xu [12]. Our first result characterizes all orderings that satisfy strong Pareto, anonymity and the strict transfer principle. They are extensions of an infinite-horizon formulation of the well-known generalized Lorenz quasi-ordering (Shorrocks [18]).

In the presence of strong Pareto, the axiom of equity preference (the infinite-horizon version of Hammond equity) is a strengthening of the strict transfer principle. We use it to identify a subclass of the class of orderings satisfying the three axioms just mentioned. These orderings are extensions of a particular infinite-horizon incomplete version of leximin. This second result leaves a larger class of orderings than that identified by Asheim and Tungodden [3] because they employ an additional axiom. The relationship between our leximin characterization and that of Asheim and Tungodden is analogous to the relationship between Basu and Mitra's [6] characterizations of infinite-horizon utilitarianism and of the overtaking criterion.

2 Basic Definitions

The set of infinite utility streams is $X = \mathbb{R}^{\mathbb{N}}$, where $\mathbb{R}$ denotes the set of all real numbers and $\mathbb{N}$ denotes the set of all natural numbers. A typical element of X is an infinite-dimensional vector $x = (x_1, x_2, \ldots, x_n, \ldots)$ and, for $n \in \mathbb{N}$, we write $x^{-n} = (x_1, \ldots, x_n)$ and $x^{+n} = (x_{n+1}, x_{n+2}, \ldots)$. The standard interpretation of $x \in X$ is that of a countably infinite utility stream where x_n is the utility experienced in period $n \in \mathbb{N}$. Of course, other interpretations are possible — for example, x_n could be the utility of an individual in a countably infinite population.

Our notation for vector inequalities on X is as follows. For all $x, y \in X$, (i) $x \geq y$ if $x_n \geq y_n$ for all $n \in \mathbb{N}$; (ii) $x > y$ if $x \geq y$ and $x \neq y$; (iii) $x \gg y$ if $x_n > y_n$ for all $n \in \mathbb{N}$. For $n \in \mathbb{N}$ and $x \in X$, $(x^{-n}_{(1)}, \ldots, x^{-n}_{(n)})$ is a rank-ordered permutation of x^{-n} such that $x^{-n}_{(1)} \leq \ldots \leq x^{-n}_{(n)}$, ties being broken arbitrarily.

$R \subseteq X \times X$ is a weak preference relation on X with strict preference $P(R)$ and indifference relation $I(R)$. A quasi-ordering is a reflexive and transitive relation, and an ordering is a complete quasi-ordering. Analogously, a partial order is an asymmetric and transitive relation, and a linear order is a complete partial order. Let R and R' be relations on X. R' is an extension of R if $R \subseteq R'$ and $P(R) \subseteq P(R')$. If an extension R' of R is an ordering, we call it an ordering extension of R, and if R' is an extension of R that is a linear order, we refer to it as a linear order extension of R.

A finite permutation of $\mathbb{N}$ is a bijection $\rho: \mathbb{N} \rightarrow \mathbb{N}$ such that there exists $m \in \mathbb{N}$ with $\rho(n) = n$ for all $n \in \mathbb{N} \setminus \{1, \ldots, m\}$. $x^\rho = (x_{\rho(1)}, x_{\rho(2)}, \ldots, x_{\rho(n)}, \ldots)$ is the finite permutation of $x \in X$ that results from relabelling the components of x in accordance with the finite permutation ρ.

Two of the most fundamental axioms in this area are the strong Pareto principle and finite anonymity, defined as follows.

Strong Pareto: For all $x, y \in X$, if $x > y$, then $(x, y) \in P(R)$.

Finite anonymity: For all $x \in X$ and for all finite permutations ρ of $\mathbb{N}$,

$$(x^\rho, x) \in I(R).$$

Szpilrajn's [23] fundamental result establishes that every partial order has a linear order extension. Arrow [1, p.64] presents a variant of Szpilrajn's theorem stating that every quasi-ordering has an ordering extension; see also Hansson [11]. This implies that the sets of orderings characterized in the theorems of the following sections are non-empty.

3 Transfer-Sensitive Infinite-Horizon Orderings

Now we examine the consequences of adding the strict transfer principle to strong Pareto and finite anonymity. A Pigou-Dalton transfer is a transfer of a positive amount of utility from a better-off agent to a worse-off agent so that the relative ranking of the two agents in the post-transfer utility stream is the same as their relative ranking in the pre-transfer stream. The strict transfer principle requires that any Pigou-Dalton transfer leads to a utility stream that is strictly preferred to the pre-transfer stream.

Strict transfer principle: For all $x, y \in X$ and for all $m, n \in \mathbb{N}$, if $x_k = y_k$ for all $k \in \mathbb{N} \setminus \{m, n\}$, $y_m > x_m \geq x_n > y_n$ and $x_n + x_m = y_n + y_m$, then $(x, y) \in P(R)$.

The strict transfer principle is the natural analogue of the corresponding condition for finite streams; see also Hara, Shinotsuka, Suzumura and Xu [12]. An alternative (equivalent) formulation of the strict transfer principle involves the explicit expression of the amount transferred from m to n when moving from y to x (this amount is $\delta = y_m - x_m = x_n - y_n$ and is readily obtained from our statement of the axiom). Although this alternative may be more standard in the literature, we use the version introduced above because it is parallel in structure to the equity-preference axioms to be defined in the following section.

To define the class of orderings satisfying the three axioms introduced thus far, we begin with a statement of Shorrocks' [18] generalized Lorenz quasi-ordering R_g^n for a society consisting of $n \in \mathbb{N}$ individuals. This quasi-ordering generalizes the standard Lorenz quasi-ordering by extending the relevant dominance criterion to comparisons involving different average (or total) utilities. For all $x, y \in X$,

$$(x^{-n}, y^{-n}) \in R_g^n \quad \Leftrightarrow \quad \sum_{i=1}^{k} x_{(i)}^{-n} \geq \sum_{i=1}^{k} y_{(i)}^{-n} \text{ for all } k \in \{1, \ldots, n\}.$$

The relation $R_G^n \subseteq X \times X$ is defined by letting, for all $x, y \in X$,

$$(x, y) \in R_G^n \quad \Leftrightarrow \quad (x^{-n}, y^{-n}) \in R_g^n \text{ and } x^{+n} \geq y^{+n}.$$

Clearly, R_G^n is a quasi-ordering for all $n \in \mathbb{N}$. The infinite-horizon extension of the generalized Lorenz quasi-ordering that is of interest in this chapter is defined by $R_G = \cup_{n \in \mathbb{N}} R_G^n$. The relation R_G can be shown to be a quasi-ordering and we characterize the class of its ordering extensions in the following theorem.

Theorem 1 *An ordering R on X satisfies strong Pareto, finite anonymity and the strict transfer principle if and only if R is an ordering extension of R_G.*

Proof. 'If.' *Step 1.* We show that the relations R_G^n and their associated strict preference relations $P(R_G^n)$ are nested, that is, for all $n \in \mathbb{N}$,

$$R_G^n \subseteq R_G^{n+1} \tag{1}$$

and

$$P(R_G^n) \subseteq P(R_G^{n+1}). \tag{2}$$

To prove (1), suppose that $(x, y) \in R_G^n$. By definition, $(x^{-n}, y^{-n}) \in R_g^n$ and $x^{+n} \geq y^{+n}$ and, thus,

$$\sum_{i=1}^{k} x_{(i)}^{-n} \geq \sum_{i=1}^{k} y_{(i)}^{-n} \text{ for all } k \in \{1, \ldots, n\}, \tag{3}$$

$$x_{n+1} \geq y_{n+1} \tag{4}$$

and

$$x^{+(n+1)} \geq y^{+(n+1)}. \tag{5}$$

Because of (5), it is sufficient to prove that

$$\sum_{i=1}^{k} x_{(i)}^{-(n+1)} \geq \sum_{i=1}^{k} y_{(i)}^{-(n+1)} \quad \text{for all } k \in \{1, \ldots, n+1\}. \tag{6}$$

If $k = n + 1$, we have

$$\sum_{i=1}^{n+1} x_{(i)}^{-(n+1)} = \sum_{i=1}^{n} x_{(i)}^{-n} + x_{n+1}$$

and

$$\sum_{i=1}^{n+1} y_{(i)}^{-(n+1)} = \sum_{i=1}^{n} y_{(i)}^{-n} + y_{n+1}.$$

Adding (3) for $k = n$ and (4), we obtain (6) for $k = n + 1$.

Now let $k \in \{1, \ldots, n\}$. We distinguish the following four cases which cover all possibilities.

Case 1. $x_{n+1} \geq x_{(k)}^{-n}$ and $y_{n+1} \geq y_{(k)}^{-n}$. This implies $x_{(i)}^{-(n+1)} = x_{(i)}^{-n}$ and $y_{(i)}^{-(n+1)} = y_{(i)}^{-n}$ for all $i \in \{1, \ldots, k\}$, and (6) for this k follows immediately from (3).

Case 2. $x_{n+1} \leq x_{(k)}^{-n}$ and $y_{n+1} \leq y_{(k)}^{-n}$. This implies

$$\sum_{i=1}^{k} x_{(i)}^{-(n+1)} = \sum_{i=1}^{k-1} x_{(i)}^{-n} + x_{n+1}$$

and

$$\sum_{i=1}^{k} y_{(i)}^{-(n+1)} = \sum_{i=1}^{k-1} y_{(i)}^{-n} + y_{n+1}.$$

Adding (3) and (4), we obtain (6) for this k.

Case 3. $x_{n+1} < x_{(k)}^{-n}$ and $y_{n+1} > y_{(k)}^{-n}$. This implies

$$\sum_{i=1}^{k} x_{(i)}^{-(n+1)} = \sum_{i=1}^{k-1} x_{(i)}^{-n} + x_{n+1}$$

and

$$\sum_{i=1}^{k} y_{(i)}^{-(n+1)} = \sum_{i=1}^{k-1} y_{(i)}^{-n} + y_{(k)}^{-n}.$$

Combining (4) and the inequality $y_{n+1} > y_{(k)}^{-n}$ (which is valid by definition of the present case), it follows that $x_{n+1} \geq y_{(k)}^{-n}$. Adding this inequality and (3), we obtain (6) for this k.

Case 4. $x_{n+1} > x_{(k)}^{-n}$ and $y_{n+1} < y_{(k)}^{-n}$. This implies

$$\sum_{i=1}^{k} x_{(i)}^{-(n+1)} = \sum_{i=1}^{k-1} x_{(i)}^{-n} + x_{(k)}^{-n}$$

and

$$\sum_{i=1}^{k} y_{(i)}^{-(n+1)} = \sum_{i=1}^{k-1} y_{(i)}^{-n} + y_{n+1}.$$

The inequality $y_{n+1} < y_{(k)}^{-n}$ (which is satisfied by definition of the present case) implies

$$\sum_{i=1}^{k-1} y_{(i)}^{-n} + y_{(k)}^{-n} \geq \sum_{i=1}^{k-1} y_{(i)}^{-n} + y_{n+1}.$$

Combining this inequality with (3) yields (6) for this k.

To establish (2), suppose that $(x, y) \in P(R_G^n)$. By definition, at least one of the following two statements is true:

$$(x^{-n}, y^{-n}) \in P(R_g^n) \quad \text{and} \quad x^{+n} \geq y^{+n}; \tag{7}$$
$$(x^{-n}, y^{-n}) \in R_g^n \quad \text{and} \quad x^{+n} > y^{+n}. \tag{8}$$

If (7) is true, it follows that the inequalities in (3) are satisfied and at least one of them is strict. Now $(x, y) \in P(R_G^{n+1})$ follows from noting that, in all cases distinguished in the proof of (1), the presence of a strict inequality in (3) yields (6) with at least one strict inequality.

If (8) is true, it follows as in the proof of (1) that the inequalities in (6) are satisfied. If $x_{n+1} > y_{n+1}$, it follows immediately that one of these inequalities must be strict and, together with $x^{+(n+1)} \geq y^{+(n+1)}$, we obtain $(x, y) \in P(R_G^{n+1})$. If $x_{n+1} = y_{n+1}$, we must have $x^{+(n+1)} > y^{+(n+1)}$ which, together with (6), establishes that $(x, y) \in P(R_G^{n+1})$.

Step 2. We now show that, for all $x, y \in X$,

$$(x, y) \in P(R_G) \Leftrightarrow \exists n \in \mathbb{N} \text{ such that } (x, y) \in P(R_G^n). \tag{9}$$

Suppose first that $(x, y) \in P(R_G)$. By definition, there exists $n \in \mathbb{N}$ such that $(x, y) \in R_G^n$. Moreover, $(y, x) \notin R_G^n$ because otherwise we obtain $(y, x) \in R_G$ by definition and thus a contradiction to our hypothesis that $(x, y) \in P(R_G)$. Hence $(x, y) \in P(R_G^n)$.

Conversely, suppose that there exists $n \in \mathbb{N}$ such that $(x, y) \in P(R_G^n)$. Suppose there exists $m \in \mathbb{N}$ such that $(y, x) \in R_G^m$. Clearly, $m \neq n$; otherwise we immediately obtain a contradiction. If $m > n$, $(x, y) \in P(R_G^n)$ and (repeated if necessary) application of (2) together imply $(x, y) \in P(R_G^m)$, contradicting the assumption $(y, x) \in R_G^m$. If $m < n$, $(y, x) \in R_G^m$ and (repeated if necessary) application of (1) together imply $(y, x) \in R_G^n$, contradicting the hypothesis $(x, y) \in P(R_G^n)$. We conclude that $(x, y) \in R_G^n$ and $(y, x) \notin R_G^m$ for all $m \in \mathbb{N}$. By definition, this implies $(x, y) \in P(R_G)$.

Step 3. Next, we prove that R_G is a quasi-ordering. Reflexivity is immediate because, for all $x \in X$, $(x, x) \in R_G^n$ for all $n \in \mathbb{N}$ and hence $(x, x) \in R_G$. To prove that R_G is transitive, suppose that $(x, y), (y, z) \in R_G$. By definition, there exist $m, n \in \mathbb{N}$ such that $(x, y) \in R_G^n$ and $(y, z) \in R_G^m$. Let $k = \max\{m, n\}$. By (repeated if necessary) application of (1), $(x, y), (y, z) \in R_G^k$ and by the transitivity of R_G^k, $(x, z) \in R_G^k$ which, in turn, implies $(x, z) \in R_G$.

Step 4. Now let R be an ordering extension of R_G. We complete the proof of the 'if' part by showing that R satisfies the required axioms.

To establish that strong Pareto is satisfied, suppose that $x, y \in X$ are such that $x > y$. Let $n = \min \{m \in \mathbb{N} \mid x_m > y_m\}$. By definition, $(x, y) \in P(R_G^n)$. By (9), $(x, y) \in P(R_G)$ and, because R is an ordering extension of R_G, we obtain $(x, y) \in P(R)$.

Next, we show that finite anonymity is satisfied. Let $x \in X$ and let ρ be a finite permutation of $\mathbb{N}$. By definition, there exists $m \in \mathbb{N}$ such that $\rho(n) = n$ for all $n \in \mathbb{N} \setminus \{1, \ldots, m\}$. By definition of R_G^m, $(x^\rho, x) \in I(R_G^m)$. By definition of R_G, this implies $(x^\rho, x) \in I(R_G)$. Because R is an ordering extension of R_G, we obtain $(x^\rho, x) \in I(R)$.

Finally, we show that the strict transfer principle is satisfied. Consider $x, y \in X$ and $m, n \in \mathbb{N}$ such that $x_k = y_k$ for all $k \in \mathbb{N} \setminus \{m, n\}$, $y_m > x_m \geq x_n > y_n$ and $x_n + x_m = y_n + y_m$. Let $j = \max \{m, n\}$. By definition of R_G^j, we obtain $(x, y) \in R_G^j$. By (9), $(x, y) \in R_G$ and, because R is an ordering extension of R_G, $(x, y) \in R$.

'Only if.' Suppose R is an ordering on X satisfying the three axioms of the theorem statement. To prove that R is an ordering extension of R_G, we have to establish the set inclusions $R_G \subseteq R$ and $P(R_G) \subseteq P(R)$.

Suppose $x, y \in X$ are such that $(x, y) \in R_G$. By definition, there exists $n \in \mathbb{N}$ such that

$$\sum_{i=1}^{k} x_{(i)}^{-n} \geq \sum_{i=1}^{k} y_{(i)}^{-n} \text{ for all } k \in \{1, \ldots, n\}$$

and $x^{+n} \geq y^{+n}$. By anonymity, we can without loss of generality assume that $x_{(i)}^{-n} = x_i$ and $y_{(i)}^{-n} = y_i$ for all $i \in \{1, \ldots, n\}$. Employing an argument analogous to that used by Shorrocks [18, Theorem 2], we let $w \in X$ be such that $w_j = y_j$ for all $j \in \{1, \ldots, n-1\}$, $w_n = y_n + \sum_{i=1}^n x_i - \sum_{i=1}^n y_i$ and $w^{+n} = x^{+n}$. We have $w \geq y$ and thus $(w, y) \in R$ by reflexivity (if $w = y$) or by strong Pareto (if $w > y$). Furthermore, $(x^{-n}, w^{-n}) \in R_g^n$ and $\sum_{i=1}^n x_i = \sum_{i=1}^n w_i$. If $x^{-n} = w^{-n}$, $(x, w) \in R$ follows from reflexivity (note that $x^{+n} = w^{+n}$ by definition). If $x^{-n} \neq w^{-n}$, it follows that x^{-n} can be reached from w^{-n} through a finite sequence of Pigou-Dalton transfers (see Hardy, Littlewood and Pólya [13]). Thus, by (repeated if necessary) application of the strict transfer principle (and transitivity if necessary), we obtain $(x, w) \in R$ (note again that $x^{+n} = w^{+n}$). Transitivity now implies $(x, y) \in R$.

Now let $x, y \in X$ be such that $(x, y) \in P(R_G)$. Because $P(R_G) \subseteq R_G$ by definition and $R_G \subseteq R$ as just established, it follows that $(x, y) \in R$. If $(y, x) \in R$, there exists $m \in \mathbb{N}$ such that $(y, x) \in R_G^m$. By (9), there exists $n \in \mathbb{N}$ such that $(x, y) \in P(R_G^n)$. We now obtain a contradiction using the same argument as in the proof of (9) and, thus, the hypothesis $(y, x) \in R$ must be false. Together with $(x, y) \in R$, it follows that $(x, y) \in P(R)$. ■

4 Infinite-Horizon Leximin

An equity property that has received a considerable amount of attention in finite settings is the Hammond equity and some of its variations. The infinite-horizon version we use is defined as follows.

Equity preference: For all $x, y \in X$ and for all $m, n \in \mathbb{N}$, if $x_k = y_k$ for all $k \in \mathbb{N} \setminus \{m, n\}$ and $y_m > x_m > x_n > y_n$, then $(x, y) \in R$.

Equity preference is the extension of Hammond's [10] equity axiom to the infinite-horizon environment. The axiom is used in Asheim and Tungodden [3]; see also Asheim, Mitra and Tungodden [4] for an alternative version which they call the Hammond equity for the future. A condition which is stronger than Hammond's equity axiom is used by d'Aspremont and Gevers [8] who require $(x, y) \in P(R)$ rather than merely $(x, y) \in R$ in the conclusion of the axiom. In the presence of strong Pareto, the two axioms are equivalent. Moreover, strong Pareto and equity preference together imply the following property which, in turn, obviously implies the strict transfer principle.

Strict equity preference: For all $x, y \in X$ and for all $m, n \in \mathbb{N}$, if $x_k = y_k$ for all $k \in \mathbb{N} \setminus \{m, n\}$ and $y_m > x_m \geq x_n > y_n$, then $(x, y) \in P(R)$.

To see that strict equity preference is implied by strong Pareto and equity preference, suppose that R satisfies the first two axioms, and let $x, y \in X$ and $m, n \in \mathbb{N}$ be such that $x_k = y_k$ for all $k \in \mathbb{N} \setminus \{m, n\}$ and $y_m > x_m \geq x_n > y_n$. Let $z \in X$ be such that $z_k = x_k = y_k$ for all $k \in \mathbb{N} \setminus \{m, n\}$ and $x_n > z_m > z_n > y_n$. By strong Pareto, $(x, z) \in P(R)$ and by equity preference, $(z, y) \in R$. Thus, transitivity implies $(x, y) \in P(R)$ and strict equity preference is satisfied.

If the strict transfer principle is replaced by equity preference (which, in the presence of strong Pareto, is a strengthening), the only remaining orderings are infinite-horizon versions of the leximin criterion. For each $n \in \mathbb{N}$, we denote the usual leximin ordering on $\mathbb{R}^n$ by R^n_ℓ, that is, for all $x, y \in X$,

$$(x^{-n}, y^{-n}) \in R^n_\ell \quad \Leftrightarrow \quad x^{-n} \text{ is a permutation of } y^{-n} \text{ or there exists } m \in \{1, \ldots, n\} \text{ such that } x^{-n}_{(k)} = y^{-n}_{(k)} \text{ for all } k \in \{1, \ldots, n\} \setminus \{m, \ldots, n\} \text{ and } x^{-n}_{(m)} > y^{-n}_{(m)}.$$

Again, let $n \in \mathbb{N}$ and define a relation $R^n_L \subseteq X \times X$ by letting, for all $x, y \in X$,

$$(x, y) \in R^n_L \quad \Leftrightarrow \quad (x^{-n}, y^{-n}) \in R^n_\ell \text{ and } x^{+n} \geq y^{+n}.$$

This relation can be shown to be a quasi-ordering for all $n \in \mathbb{N}$. Finally, let $R_L = \cup_{n \in \mathbb{N}} R^n_L$. This relation is a quasi-ordering but it is not complete — some infinite utility streams are not ranked by R_L. Our next result characterizes all ordering extensions of R_L.

Theorem 2 *An ordering R on X satisfies strong Pareto, finite anonymity and equity preference if and only if R is an ordering extension of R_L.*

Proof. 'If.' As in the proof of Theorem 1, we begin by showing that the relations R^n_L and their associated strict preference relations $P(R^n_L)$ are nested, that is, for all $n \in \mathbb{N}$,

$$R^n_L \subseteq R^{n+1}_L \tag{10}$$

and

$$P(R_L^n) \subseteq P(R_L^{n+1}). \tag{11}$$

To prove (10), suppose that $(x, y) \in R_L^n$. By definition, $(x^{-n}, y^{-n}) \in R_\ell^n$ and $x^{+n} \geq y^{+n}$. Then, either x^{-n} is a permutation of y^{-n} and $x^{+n} \geq y^{+n}$, or there exists $j \in \{1, \ldots, n\}$ such that $x_{(k)}^{-n} = y_{(k)}^{-n}$ for all $k \in \{1, \ldots, n\} \setminus \{j, \ldots, n\}$, $x_{(j)}^{-n} > y_{(j)}^{-n}$ and $x^{+n} \geq y^{+n}$. In both cases, $(x^{-(n+1)}, y^{-(n+1)}) \in R_\ell^{n+1}$ and $x^{+(n+1)} \geq y^{+(n+1)}$, that is, $(x, y) \in R_L^{n+1}$.

To establish (11), suppose that $(x, y) \in P(R_L^n)$. By definition, at least one of the following two statements is true:

$$(x^{-n}, y^{-n}) \in P(R_\ell^n) \quad \text{and} \quad x^{+n} \geq y^{+n}; \tag{12}$$

$$(x^{-n}, y^{-n}) \in R_\ell^n \quad \text{and} \quad x^{+n} > y^{+n}. \tag{13}$$

By (10), it follows that $(x, y) \in R_L^{n+1}$. To prove that $(x, y) \in P(R_L^{n+1})$, suppose, by way of contradiction, that $(y, x) \in R_L^{n+1}$. Then, by definition,

$$(x^{-n}, y^{-n}) \in I(R_\ell^n) \text{ and } x^{+n} = y^{+n},$$

contradicting (12) and (13).

Using the same arguments as in the proof of (9) in Theorem 1 (replacing R_G and R_G^n with R_L and R_L^n), it follows that, for all $x, y \in X$,

$$(x, y) \in P(R_L) \Leftrightarrow \exists n \in \mathbb{N} \text{ such that } (x, y) \in P(R_L^n) \tag{14}$$

and, furthermore, that R_L is a quasi-ordering and that any ordering extension of R_L satisfies strong Pareto and finite anonymity.

We complete the proof of the 'if' part by showing that any ordering extension R of R_L satisfies equity preference. Consider $x, y \in X$ and $m, n \in \mathbb{N}$ such that $x_k = y_k$ for all $k \in \mathbb{N} \setminus \{m, n\}$ and $y_m > x_m > x_n > y_n$. Let $j = \max\{m, n\}$. By definition of R_L^j, we obtain $(x, y) \in R_L^j$. By (14), $(x, y) \in R_L$ and, because R is an ordering extension of R_L, $(x, y) \in R$.

'Only if.' Suppose R is an ordering on X satisfying the three axioms of the theorem statement. Fix $n \in \mathbb{N}$ and $z \in X$ and define the relation $Q^n(z) \subseteq \mathbb{R}^n \times \mathbb{R}^n$ as follows. For all $x, y \in X$,

$$(x^{-n}, y^{-n}) \in Q^n(z) \Leftrightarrow ((x^{-n}, z^{+n}), (y^{-n}, z^{+n})) \in R.$$

$Q^n(z)$ is an ordering because R is. Furthermore, it is clear that

$$(x^{-n}, y^{-n}) \in P(Q^n(z)) \Leftrightarrow ((x^{-n}, z^{+n}), (y^{-n}, z^{+n})) \in P(R) \tag{15}$$

for all $x, y \in X$. The three axioms imply that $Q^n(z)$ must satisfy the n-person versions of the axioms and, using Hammond's [10, Theorem 7.2] characterization of n-person leximin (see also d'Aspremont and Gevers [8, Theorem 5]), it follows that

$$Q^n(z) = R_\ell^n. \tag{16}$$

Because n and z were chosen arbitrarily, (16) is true for all $n \in \mathbb{N}$ and for any $z \in X$.

To prove that R is an ordering extension of R_L, we first establish the set inclusion $R_L \subseteq R$. Suppose that $x, y \in X$ are such that $(x, y) \in R_L$. By definition of R_L, there exists $n \in \mathbb{N}$ such that $(x, y) \in R_L^n$, that is,

$$(x^{-n}, y^{-n}) \in R_\ell^n \text{ and } x^{+n} \geq y^{+n}.$$

Hence, by (16),

$$(x^{-n}, y^{-n}) \in Q^n(z) \text{ and } x^{+n} \geq y^{+n}$$

for all $z \in X$. Choosing $z = y$ and using the definition of $Q^n(z)$, it follows that $((x^{-n}, y^{+n}), (y^{-n}, y^{+n})) \in R$. Because $x^{+n} \geq y^{+n}$, reflexivity (if $x^{+n} = y^{+n}$) or the conjunction of strong Pareto and transitivity (if $x^{+n} > y^{+n}$) implies $((x^{-n}, x^{+n}), (y^{-n}, y^{+n})) = (x, y) \in R$.

We complete the proof by establishing the set inclusion $P(R_L) \subseteq P(R)$. Let $x, y \in X$ be such that $(x, y) \in P(R_L)$. By (14), there exists $n \in \mathbb{N}$ such that $(x, y) \in P(R_L^n)$. Thus, (12) or (13) is true.

If (12) holds, (16) implies

$$(x^{-n}, y^{-n}) \in P(Q^n(z)) \text{ and } x^{+n} \geq y^{+n}$$

for all $z \in X$. Setting $z = y$ and using (15), we obtain $((x^{-n}, y^{+n}), (y^{-n}, y^{+n})) \in P(R)$ and, using reflexivity or strong Pareto and transitivity as in the proof of the set inclusion $R_L \subseteq R$, we obtain $(x, y) \in P(R)$.

If (13) holds, (16) implies

$$(x^{-n}, y^{-n}) \in Q^n(z) \text{ and } x^{+n} > y^{+n}$$

for all $z \in X$. Setting $z = y$, it follows that $((x^{-n}, y^{+n}), (y^{-n}, y^{+n})) \in R$ as a consequence of the definition of $Q^n(z)$ and, by strong Pareto and transitivity, $((x^{-n}, x^{+n}), (y^{-n}, y^{+n})) = (x, y) \in P(R)$. ■

5 Concluding Remarks

The results of this chapter reinforce the findings of earlier contributions regarding the existence of orderings of infinite utility streams with attractive properties. In particular, we provide characterizations of two classes of such orderings. Given the existential nature of the proofs, we do not provide explicit constructions of these orderings. However, this feature is by no means unique to our approach. Extending quasi-orderings to orderings often requires non-constructive techniques; see, for example, Richter's [17] use of Szpilrajn's [23] extension theorem in the context of rational choice.

A plausible conclusion to be drawn is that impossibility results such as those of Diamond [9], Basu and Mitra [5] and Hara, Shinotsuka, Suzumura and Xu [12] can be avoided if continuity or representability assumptions are dispensed with. Because continuity and representability can be considered rather demanding in infinite-horizon settings, this confirms, in our view, that the state of affairs in this area is not as disappointing and negative as has been suggested by the impossibility results of many earlier contributions.

The techniques employed to characterize infinite-horizon versions of the generalized-Lorenz criterion and of leximin appear to be very powerful and applicable to the extension of other finite-population social-choice rules; see also the characterization of infinite-horizon utilitarianism by Basu and Mitra [6]. We hope that our approach will stimulate further research in the area of intergenerational social choice by identifying alternative sets of attractive axioms and characterizing the social orderings that satisfy them.

The classes of orderings characterized in this chapter are relatively large: there are many comparisons of utility streams that are not determined by the axioms employed. An issue to be addressed in future work is to examine to what extent the ranking of more pairs of streams can be determined by employing plausible additional axioms.

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Chapter 8
Infinite-Horizon Choice Functions*

1 Introduction

The literature on ranking infinite consumption (or utility) streams has produced a number of negative results in the form of the incompatibility of seemingly mild axioms. For example, following Koopmans [15], Diamond [11] establishes that anonymity is incompatible with the strong Pareto principle. Moreover, he shows that if anonymity is weakened to *finite* anonymity — which restricts the application of the standard anonymity requirement to situations where utility streams differ in at most a finite number of components — and a continuity requirement is added, an impossibility results again. Hara, Shinotsuka, Suzumura and Xu [14] adapt the well-known strict transfer principle due to Pigou [17] and Dalton [10] to the infinite-horizon context. They show that this principle is incompatible with strong Pareto and continuity even if the social preference is merely required to be acyclical. Basu and Mitra [5] show that strong Pareto, finite anonymity and representability by a real-valued function are incompatible. Epstein [12] establishes the incompatibility of a set of standard axioms and a substitution property requiring the possibility to improve upon any given constant stream by means of a stream with lower initial consumption.

The main purpose of this chapter is to suggest an alternative approach that may provide a promising way to address issues involving intergenerational allocation problems with an infinite horizon. Instead of searching for a *ranking* of infinite streams, we examine a choice-theoretic model where a *choice function* is used to select a consumption stream from each set of feasible streams. Because our focus is on the choice-theoretic aspect of the model, we deliberately consider a simple setting where there is a single resource and a linear and stationary technology with positive renewal. This implies that the feasibility of a consumption stream is determined by the initial amount of the resource available, and the choice function assigns a consumption stream (the chosen consumption stream, given the feasibility constraint) to each possible initial amount.

We begin with an analysis of two fundamental properties whose versions formulated for orderings have been used extensively in the literature, namely, *efficiency* and *time con-*

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sistency. We provide characterizations of all infinite-horizon choice functions satisfying either of the two axioms and, moreover, identify all choice functions with both properties. We then consider *equity* properties that are choice-theoretic versions of the Suppes-Sen principle, the Pigou-Dalton transfer principle and resource monotonicity (see Asheim and Tungodden [3]; Bossert, Sprumont and Suzumura [8]; Hara, Shinotsuka, Suzumura and Xu [14], for equity properties imposed on rankings of infinite streams). Again, classes of infinite-horizon choice functions possessing one of these properties are characterized, and further axiomatizations are obtained by adding efficiency or time consistency.

The results we obtain are promising. Unlike in the case of orderings of infinite utility streams, impossibilities can be avoided and rich classes of infinite-horizon choice functions satisfying several desirable properties do exist. In particular, our choice-theoretic version of the Suppes-Sen principle imposes full anonymity rather than merely finite anonymity and our choice functions may be continuous in the initial endowment. Moreover, it turns out that the notion of *sustainability,* which has played a major role in the literature on intergenerational resource allocation, is closely linked to the Suppes-Sen and Pigou-Dalton principles. Our conclusion from these results is that the choice-theoretic approach to intergenerational resource allocation provides an interesting and viable alternative to the models based on establishing orderings of infinite utility streams, and we propose to explore this approach further.

Section 2 contains some basic definitions and a first well-known observation characterizing sets of feasible consumption streams. In Section 3, we examine the fundamental axioms of efficiency and time consistency. We characterize all efficient infinite-horizon choice functions, all time-consistent infinite-horizon choice functions, and the class of choice functions satisfying both requirements. Section 4 deals with the equity axioms à la Suppes-Sen, Pigou-Dalton and resource monotonicity. We characterize all infinite-horizon choice functions satisfying: (i) Suppes-Sen; (ii) efficiency and Pigou-Dalton; (iii) time consistency and Suppes-Sen; (iv) efficiency, time consistency and Pigou-Dalton; (v) efficiency, time consistency and resource monotonicity. As a by-product of our analysis, we show that the conjunction of efficiency and Pigou-Dalton is equivalent to Suppes-Sen. Section 5 provides some examples and Section 6 concludes.

2 Preliminaries

Let $\mathbb{R}_+$ and $\mathbb{R}_{++}$ denote the set of all non-negative real numbers and the set of all positive real numbers, respectively. Analogously, $\mathbb{Z}_+$ and $\mathbb{Z}_{++}$ denote the set of all non-negative integers and the set of all positive integers, respectively.

The set $\mathcal{Y} = \mathbb{R}_+^{\mathbb{Z}_+}$ is defined to be the set of all sequences $y = (y_0, y_1, \ldots, y_t, \ldots)$. We interpret y as a consumption stream, where y_t is the amount of a single resource consumed in period $t \in \mathbb{Z}_+$. Time is measured relative to the present: period t is the t^{th} period after today. The initial amount of the resource is $x \in \mathbb{R}_+$. We assume a linear and stationary technology, entailing that in each period, the resource is renewed at the fixed positive rate $r \in \mathbb{R}_{++}$.

We use the following notation for inequalities in $\mathcal{Y}$. For all $y, z \in \mathcal{Y}$, $y \geq z$ if and only if $y_t \geq z_t$ for all $t \in \mathbb{Z}_+$, and $y > z$ if and only if $y \geq z$ and $y \neq z$.

For $x \in \mathbb{R}_+$ and $y \in \mathcal{Y}$, the sequence of resource stocks

$$k(x, y) = (k_0(x, y), k_1(x, y), \ldots, k_t(x, y), \ldots) \in \mathbb{R}^{\mathbb{Z}_+}$$

generated by x and y is defined by $k_0(x, y) = x$ and

$$k_t(x, y) = (1 + r)(k_{t-1}(x, y) - y_{t-1})$$

for all $t \in \mathbb{Z}_{++}$. For $x \in \mathbb{R}_+$, the set of x-feasible consumption streams is

$$\mathcal{S}(x) = \{y \in \mathcal{Y} \mid y_t \in [0, k_t(x, y)] \text{ for all } t \in \mathbb{Z}_+\}.$$

The following well-known lemma completely characterizes $\mathcal{S}(x)$. Because the result is well-known, we state it without a proof.

Lemma 1 *For all $x \in \mathbb{R}_+$,*

$$\mathcal{S}(x) = \Big\{ y \in \mathcal{Y} \;\Big|\; \sum_{t=0}^{\infty} \frac{y_t}{(1+r)^t} \leq x \Big\}.$$

3 Efficient and Time-Consistent Choice

An *infinite-horizon choice function* is a mapping $C \colon \mathbb{R}_+ \to \mathcal{Y}$ such that $C(x) \in \mathcal{S}(x)$ for all $x \in \mathbb{R}_+$. This function assigns a consumption stream to any given initial amount of a single resource available in the economy. Note that consumption streams are undated: whether the choice takes place today or tomorrow makes no difference if the same initial endowment is present. This time-independence feature of a choice function ensures that the choice of a starting period is irrelevant. It can be motivated in terms of an equal-treatment property applied to generations. For all $t \in \mathbb{Z}_+$, we write $C_t(x)$ for the t^{th} component of the sequence $C(x)$.

The first fundamental property of an infinite-horizon choice function is the familiar *efficiency* axiom. It requires that no x-feasible consumption stream Pareto dominates the chosen consumption stream with initial stock x.

Efficiency. For all $x \in \mathbb{R}_+$ and for all $y \in \mathcal{Y}$,

$$y > C(x) \;\; \Rightarrow \;\; y \notin \mathcal{S}(x).$$

Given Lemma 1, it is straightforward to characterize the class of efficient choice functions. We omit the immediate proof of the following lemma stating the relevant result.

Lemma 2 *An infinite-horizon choice function C satisfies efficiency if and only if*

$$\sum_{t=0}^{\infty} \frac{C_t(x)}{(1+r)^t} = x \quad \textit{for all } x \in \mathbb{R}_+. \tag{1}$$

Time consistency prevents deviations from chosen consumption streams as time progresses. Thus, for any $x \in \mathbb{R}_+$ and for any $t, \tau \in \mathbb{Z}_+$, the consumption $C_{t+\tau}(x)$ in period $t+\tau$ for the initial endowment x should be the same as the consumption $C_\tau(k_t(x, C(x)))$ in period τ for the initial endowment $k_t(x, C(x))$.

Time consistency. For all $x \in \mathbb{R}_+$ and for all $t, \tau \in \mathbb{Z}_+$,

$$C_{t+\tau}(x) = C_\tau(k_t(x, C(x))).$$

We now characterize all infinite-horizon choice functions satisfying time consistency. In order to express this class of choice functions, we use a function $g \colon \mathbb{R}_+ \to \mathbb{R}_+$ that indicates, for each initial level of the resource, the amount of the resource that is available in the next period after the present consumption has taken place. Hence, we may refer to g as the *inheritance function.* Consequently, $g(x)/(1+r)$ is the bequest that is left behind, and $x - (g(x)/(1+r))$ is the present consumption. Hence, we may refer to the mapping $x \mapsto x - (g(x)/(1+r))$ as the *consumption function.*

For any function $g \colon \mathbb{R}_+ \to \mathbb{R}_+$, let the function $g^0 \colon \mathbb{R}_+ \to \mathbb{R}_+$ be defined by $g^0(x) = x$ for all $x \in \mathbb{R}_+$ and, for all $t \in \mathbb{Z}_{++}$, define the function $g^t \colon \mathbb{R}_+ \to \mathbb{R}_+$ by letting $g^t(x) = g(g^{t-1}(x))$ for all $x \in \mathbb{R}_+$. As will become clear once our characterization of time consistency is stated, the functions g^t have a natural interpretation: they identify the amount of the resource available in period t as a function of the initial endowment x only. Because all these functions are determined once a function g is chosen, it is sufficient to specify, for any initial endowment, the amount of the resource remaining at the beginning of period one.

The following lemma characterizes all time-consistent choice functions.

Lemma 3 *An infinite-horizon choice function C satisfies time consistency if and only if there exists a function $g \colon \mathbb{R}_+ \to \mathbb{R}_+$ such that*

$$g(x) \le x(1+r) \quad \textit{for all } x \in \mathbb{R}_+ \tag{2}$$

and

$$C_t(x) = g^t(x) - \frac{g^{t+1}(x)}{1+r} \quad \textit{for all } t \in \mathbb{Z}_+ \textit{ and for all } x \in \mathbb{R}_+. \tag{3}$$

Proof. 'If.' Let C be an infinite-horizon choice function and suppose there exists a function $g \colon \mathbb{R}_+ \to \mathbb{R}_+$ such that (2) and (3) are satisfied. Let $x \in \mathbb{R}_+$ and $t \in \mathbb{Z}_+$. By (2), it follows that

$$g^{t+1}(x) = g(g^t(x)) \le g^t(x)(1+r)$$

and, together with (3), that

$$C_t(x) = g^t(x) - \frac{g^{t+1}(x)}{1+r} \ge 0.$$

Using (3) and the definition of $k(x, y)$, we obtain

$$k_t(x, C(x)) = g^t(x). \tag{4}$$

Because g is non-negative-valued, (3) and (4) together imply

$$C_t(x) = g^t(x) - \frac{g^{t+1}(x)}{1+r} = k_t(x, C(x)) - \frac{g^{t+1}(x)}{1+r} \le k_t(x, C(x)).$$

Hence, $C(x) \in \mathcal{S}(x)$ and C is a well-defined infinite-horizon choice function.

To establish time consistency, let $x \in \mathbb{R}_+$ and $t, \tau \in \mathbb{Z}_+$. By (3),

$$C_{t+\tau}(x) = g^{t+\tau}(x) - \frac{g^{t+\tau+1}(x)}{1+r}. \tag{5}$$

By (4) and (3),

$$C_\tau(k_t(x, C(x))) = C_\tau(g^t(x)) = g^\tau(g^t(x)) - \frac{g^{\tau+1}(g^t(x))}{1+r} = g^{t+\tau}(x) - \frac{g^{t+\tau+1}(x)}{1+r}$$

which, together with (5), proves that C is time consistent.

'Only if.' Suppose C is an infinite-horizon choice function that satisfies time consistency. Define the function $g\colon \mathbb{R}_+ \to \mathbb{R}_+$ by letting

$$g(x) = (1+r)(x - C_0(x)) \tag{6}$$

for all $x \in \mathbb{R}_+$. By feasibility, $C_0(x) \in [0, x]$, and the definition of g immediately implies $g(x) \in [0, x(1+r)]$ for all $x \in \mathbb{R}_+$, establishing that g indeed maps into $\mathbb{R}_+$ and that (2) is satisfied.

It remains to be shown that (3) is satisfied. We proceed by induction. Solving (6) for $C_0(x)$, we obtain

$$C_0(x) = x - \frac{g(x)}{1+r} = g^0(x) - \frac{g^1(x)}{1+r}. \tag{7}$$

Now suppose

$$C_t(x) = g^t(x) - \frac{g^{t+1}(x)}{1+r} \tag{8}$$

for some $t \in \mathbb{Z}_+$. By definition, $k_1(x, C(x)) = (1+r)(x - C_0(x)) = g(x)$. Thus, using time consistency and (8), we obtain

$$C_{t+1}(x) = C_t(k_1(x, C(x)) = C_t(g(x)) = g^t(g(x)) - \frac{g^{t+1}(g(x))}{1+r} = g^{t+1}(x) - \frac{g^{t+2}(x)}{1+r}$$

which completes the proof. ■

We now characterize all infinite-horizon choice functions satisfying both efficiency and time consistency.

Theorem 1 *An infinite-horizon choice function C satisfies efficiency and time consistency if and only if there exists a function $g\colon \mathbb{R}_+ \to \mathbb{R}_+$ such that (2), (3) and*

$$\lim_{t\to\infty} \frac{g^t(x)}{(1+r)^t} = 0 \quad \textit{for all } x \in \mathbb{R}_+ \tag{9}$$

are satisfied.

Proof. 'If.' Let C be an infinite-horizon choice function and suppose there exists a function $g\colon \mathbb{R}_+ \to \mathbb{R}_+$ such that (2), (3) and (9) are satisfied. Then, by Lemma 3, C is a well-defined infinite-horizon choice function that satisfies time consistency. Furthermore,

$$\sum_{t=0}^{\infty} \frac{C_t(x)}{(1+r)^t} = x - \lim_{t\to\infty} \frac{g^t(x)}{(1+r)^t}.$$

By invoking Lemma 2, (9) implies that C satisfies efficiency.

'Only if.' Suppose C is an infinite-horizon choice function that satisfies efficiency and time consistency. Then, by Lemma 3, there exists a function $g\colon \mathbb{R}_+ \to \mathbb{R}_+$ such that (2) and (3) are satisfied. By invoking Lemma 2, efficiency and (3) imply that

$$x = \sum_{t=0}^{\infty} \frac{C_t(x)}{(1+r)^t} = x - \lim_{t\to\infty} \frac{g^t(x)}{(1+r)^t}.$$

Hence, g satisfies (9). ■

If zero is eliminated as a possible value of the initial amount of the resource, it is straightforward to obtain a similar result to the above theorem. In that case, the function g has as its domain and as its range the set $\mathbb{R}_{++}$ rather than $\mathbb{R}_+$ and the weak inequality in (2) is changed to a strict inequality.

Condition (9) is of course a capital value transversality condition, which has been used to characterize efficient capital accumulation at least since Malinvuad [16].

The properties (2) and (9) of a function $g\colon \mathbb{R}_+ \to \mathbb{R}_+$ are independent, as is straightforward to verify. That (3) must be satisfied is a consequence of the time-consistency requirement, and (2) ensures that this is done without violating the resource constraints. Property (9) is required for the efficiency axiom.

4 Imposing Equity Axioms

We now examine the consequences of imposing certain equity axioms, in addition to efficiency and time consistency.

The first of the equity axioms that we consider — *Suppes-Sen* — requires that no x-feasible consumption stream has a permutation which Pareto dominates the chosen consumption stream with initial stock x. The term 'permutation' signifies a bijective mapping π of $\mathbb{Z}_+$ onto itself. The Suppes-Sen axiom is a straightforward adaptation of the Suppes-Sen principle for orderings (cf. Suppes [20]; Sen [19] to the present infinite-horizon choice-theoretic setting. Clearly, the axiom as defined below implies efficiency.

Suppes-Sen. For all $x \in \mathbb{R}_+$ and for all $y, y' \in \mathcal{Y}$, if y' is a permutation of y, then

$$y' > C(x) \quad \Rightarrow \quad y \notin \mathcal{S}(x).$$

Note that we do not restrict the scope of the axiom to finite permutations (that is, permutations π with the property that there is a $t \in \mathbb{Z}_+$ such that $\pi(\tau) = \tau$ for all $\tau \geq t$).

In contrast to the Suppes-Sen axiom formulated for orderings of infinite utility streams, allowing for infinite permutations does not lead to an impossibility in our choice-theoretic setting.

Our next result characterizes all choice functions satisfying the Suppes-Sen principle.

Lemma 4 *An infinite-horizon choice function C satisfies Suppes-Sen if and only if (1) and*

$$C_t(x) \leq C_{t+1}(x) \quad \textit{for all } x \in \mathbb{R}_+ \textit{ and for all } t \in \mathbb{Z}_+ \tag{10}$$

are satisfied.

Proof. 'If.' Assume (1) and (10) are satisfied. Since the sequence $\langle 1/(1+r)^t \rangle_{t\in\mathbb{Z}_+}$ is decreasing and the sequence $\langle C_t(x) \rangle_{t\in\mathbb{Z}_+}$ is non-decreasing, if y is a permutation of $C(x)$, then

$$\sum_{t=0}^{\infty} \frac{y_t}{(1+r)^t} \geq x.$$

Hence, for all $y, y' \in \mathcal{Y}$ such that y' is a permutation of y, $y' > C(x)$ implies

$$\sum_{t=0}^{\infty} \frac{y_t}{(1+r)^t} > x.$$

By Lemma 1, $y \notin \mathcal{S}(x)$. Thus, C satisfies Suppes-Sen.

'Only if.' Let $x \in \mathbb{R}_+$. Suppose first that $\sum_{t=0}^{\infty} C_t(x)/(1+r)^t < x$. Then by Lemma 1, there exists $y \in \mathcal{S}(x)$ such that $y > C(x)$. Thus, there is an x-feasible consumption stream which Pareto-dominates the chosen consumption stream with initial stock x, entailing that C does not satisfy Suppes-Sen. Together with feasibility, this contradiction implies that we must have $\sum_{t=0}^{\infty} C_t(x)/(1+r)^t = x$. By way of contradiction, suppose there exists $\tau \in \mathbb{Z}_+$ such that $C_\tau(x) > C_{\tau+1}(x)$. Construct $y \in \mathcal{Y}$ as follows:

$$y_t = \begin{cases} C_t(x) & \text{if} \quad t \notin \{\tau, \tau+1\}, \\ C_{\tau+1}(x) & \text{if} \quad t = \tau, \\ C_\tau(x) + r(C_\tau(x) - C_{\tau+1}(x)) & \text{if} \quad t = \tau+1. \end{cases}$$

Then

$$\begin{aligned} \sum_{t=0}^{\infty} \frac{y_t}{(1+r)^t} &= \sum_{t\notin\{\tau,\tau+1\}} \frac{C_t(x)}{(1+r)^t} + \frac{1}{(1+r)^\tau} \left(C_{\tau+1}(x) + \frac{C_\tau(x) + r(C_\tau(x) - C_{\tau+1}(x))}{1+r} \right) \\ &= \sum_{t=0}^{\infty} \frac{C_t(x)}{(1+r)^t} = x, \end{aligned}$$

implying by Lemma 1 that $y \in \mathcal{S}(x)$. Construct $y' \in \mathcal{Y}$ from y by permuting y_τ and $y_{\tau+1}$. Since $r(C_\tau(x) - C_{\tau+1}(x)) > 0$, we have that $y' > C(x)$. Thus, there is an x-feasible consumption stream with a permutation which Pareto-dominates the chosen consumption stream with initial stock x, entailing that C does not satisfy Suppes-Sen. ■

As is apparent from the proof, the Suppes-Sen principle as stated in the theorem can be replaced with its finite counterpart, restricting its conclusion to finite permutations. In our setting, the two properties are equivalent and we chose to use the general version in order to illustrate that, unlike the model based on orderings of infinite streams, our approach does not lead to an impossibility when infinite permutations are permitted.

The second of the equity axioms — *Pigou-Dalton* — requires that no x-feasible consumption stream can be generated from the chosen consumption stream with initial stock x through a transfer of consumption from a better-off to a worse-off generation. The axiom is a straightforward adaptation of the Pigou-Dalton transfer principle (cf. Pigou [17]; Dalton [10]) for social welfare orderings to the present choice-theoretic setting.

Pigou-Dalton. For all $x \in \mathbb{R}_+$ and for all $y, y' \in \mathcal{Y}$, if there exist $\varepsilon \in \mathbb{R}_{++}$ and $\tau, \tau' \in \mathbb{Z}_+$ such that $y_\tau = y'_\tau - \varepsilon \geq y'_{\tau'} + \varepsilon = y_{\tau'}$ and $y_t = y'_t$ for all $t \in \mathbb{Z}_+ \backslash \{\tau, \tau'\}$, then

$$y' = C(x) \;\;\Rightarrow\;\; y \notin \mathcal{S}(x).$$

We now characterize all infinite-horizon choice functions satisfying efficiency and the Pigou-Dalton principle. Interestingly, this is the same class as the one identified in the previous lemma.

Lemma 5 *An infinite-horizon choice function C satisfies efficiency and Pigou-Dalton if and only if (1) and (10) are satisfied.*

Proof. 'If.' Assume (1) and (10) are satisfied. By Lemma 2, C satisfies efficiency. Since the sequence $\langle 1/(1+r)^t \rangle_{t \in \mathbb{Z}_+}$ is decreasing and the sequence $\langle C_t(x) \rangle_{t \in \mathbb{Z}_+}$ is nondecreasing, if $y_\tau = C_\tau(x) - \varepsilon \geq C_{\tau'}(x) + \varepsilon = y_{\tau'}$ for some $\varepsilon \in \mathbb{R}_{++}$ and $y_t = C_t(x)$ for all $t \in \mathbb{Z}_+ \setminus \{\tau, \tau'\}$, then

$$\sum_{t=0}^{\infty} \frac{y_t}{(1+r)^t} > x.$$

By Lemma 1, $y \notin \mathcal{S}(x)$. Thus, C satisfies Pigou-Dalton.

'Only if.' As in the proof of Lemma 4, we may verify that $\sum_{t=0}^{\infty} C_t(x) \setminus (1+r)^t = x$ must hold.

Now suppose there exists $\tau \in \mathbb{Z}_+$ such that $C_\tau(x) > C_{\tau+1}(x)$. Construct $y \in \mathcal{Y}$ as follows:

$$y_t \;=\; \begin{cases} C_t(x) & \text{if} \quad t \notin \{\tau, \tau+1\}, \\ C_\tau(x) - \varepsilon & \text{if} \quad t = \tau, \\ C_{\tau+1}(x) + \varepsilon & \text{if} \quad t = \tau + 1, \end{cases}$$

where $0 < \varepsilon \leq (C_\tau(x) - C_{\tau+1}(x))/2$, so that $y_\tau = C_\tau(x) - \varepsilon \geq C_{\tau+1}(x) + \varepsilon = y_{\tau+1}$. Then

$$\begin{aligned} \sum_{t=0}^{\infty} \frac{y_t}{(1+r)^t} &= \sum_{t \notin \{\tau, \tau+1\}} \frac{C_t(x)}{(1+r)^t} + \frac{1}{(1+r)^\tau} \left(C_\tau(x) - \varepsilon + \frac{C_{\tau+1}(x) + \varepsilon}{1+r} \right) \\ &= \sum_{t=0}^{\infty} \frac{C_t(x)}{(1+r)^t} - \frac{r\varepsilon}{(1+r)^{\tau+1}} \;<\; x, \end{aligned}$$

implying by Lemma 1 that $y \in \mathcal{S}(x)$. Thus, an x-feasible consumption stream can be generated from the chosen consumption stream with initial stock x through a transfer of consumption from a better-off to a worse-off generation, entailing that C does not satisfy Pigou-Dalton. ■

The following corollary is an immediate consequence of the previous two lemmas.

Corollary 1 *An infinite-horizon choice function C satisfies Suppes-Sen if and only if C satisfies efficiency and Pigou-Dalton.*

The following theorem identifies all choice functions satisfying time consistency in addition to Suppes-Sen (or, equivalently, in addition to efficiency and Pigou-Dalton).

Theorem 2 *An infinite-horizon choice function C satisfies time consistency and Suppes-Sen (or efficiency, time consistency and Pigou-Dalton) if and only if there exists a function $g\colon \mathbb{R}_+ \to \mathbb{R}_+$ such that (2), (3), (9),*

$$x \leq g(x) \quad \textit{for all}\, x \in \mathbb{R}_+ \tag{11}$$

and

$$x - \frac{g(x)}{1+r} \leq g(x) - \frac{g^2(x)}{1+r} \quad \textit{for all } x \in \mathbb{R}_+ \tag{12}$$

are satisfied.

Proof. 'If.' Suppose there exists a function $g\colon \mathbb{R}_+ \to \mathbb{R}_+$ such that (2), (3), (9), (11) and (12) are satisfied. By Theorem 1, C satisfies time consistency and efficiency. Thus, by Lemma 2, (1) is satisfied. By (3) and (12), it follows that

$$\begin{aligned} C_t(x) &= g^t(x) - \frac{g^{t+1}(x)}{1+r} = g^t(x) - \frac{g(g^t(x))}{1+r} \\ &\leq g(g^t(x)) - \frac{g^2(g^t(x))}{1+r} = g^{t+1}(x) - \frac{g^{t+2}(x)}{1+r} = C_{t+1}(x) \end{aligned}$$

for all $x \in \mathbb{R}_+$ and for all $t \in \mathbb{Z}_+$. Hence, by Lemma 4, C satisfies Suppes-Sen.

'Only if.' Assume that C satisfies time consistency and Suppes-Sen. By Lemma 4, (1) and (10) are satisfied and, by Lemma 2, C satisfies efficiency. By Theorem 1, there exists a function $g\colon \mathbb{R}_+ \to \mathbb{R}_+$ satisfying (2), (3) and (9).

To show (11), suppose there exists $x \in \mathbb{R}_+$ such that $x > g(x)$. By (3) and (9), it follows that

$$\sum_{t=0}^{\infty} \frac{C_t(x)}{(1+r)^t} = x > g(x) = \sum_{t=0}^{\infty} \frac{C_{t+1}(x)}{(1+r)^t},$$

contradicting (10).

To show (12), suppose there exists $x \in \mathbb{R}_+$ such that

$$x - \frac{g(x)}{1+r} > g(x) - \frac{g^2(x)}{1+r}.$$

By (3),

$$C_0(x) = x - \frac{g(x)}{1+r} > g(x) - \frac{g^2(x)}{1+r} = C_1(x),$$

again contradicting (10). ■

Condition (11) ensures sustainable development in the sense that the current consumption can potentially be shared by all future generations. In the context of a stationary technology with only one resource (or capital good), this requires that the resource stock is maintained from the current period to the next, which is just what condition (11) entails. Condition (12) complements (11) by requiring that the potential for sharing present consumption with future generation actually materializes. Hence, Theorem 2 means that both the Suppes-Sen axiom and the Pigou-Dalton axiom can be used to justify sustainability in the present choice-theoretic setting.

Theorem 2 thereby echoes similar results when infinite-horizon social choice is analyzed through social welfare relations.

- In particular, Asheim, Buchholz and Tungodden [2] show how the Suppes-Sen principle for social welfare relations can be used to rule out unsustainable consumption streams as maximal elements under technological conditions satisfied by the simple linear model considered here. This result also implies that social welfare relations like those considered in Asehim and Tungodden [3], Basu and Mitra [6], and Bossert, Sprumont and Suzumura [8], which all satisfy the Suppes-Sen principle, yield sustainable consumption streams as maximal elements as long as maximal elements exist.

- Asheim [1] shows in a similar way how the Pigou-Dalton principle for social welfare relations can be used to rule out unsustainable consumption streams.

Another equity axiom that appears to be natural in this context is *resource monotonicity*. It requires that no one should be worse off as a consequence of an increase in the initial level of the resource. See Thomson [21] for a discussion of resource monotonicity in a variety of economic models and further references. Formulated for infinite-horizon choice functions, the axiom is defined as follows.

Resource monotonicity. For all $x, x' \in \mathbb{R}_+$,

$$x > x' \ \Rightarrow \ C(x) \geq C(x').$$

Adding resource monotonicity to efficiency and time consistency leads to the choice functions characterized in the following theorem.

Theorem 3 *An infinite-horizon choice function C satisfies efficiency, time consistency and resource monotonicity if and only if there exists a function $g\colon \mathbb{R}_+ \to \mathbb{R}_+$ such that (2), (3), (9),*

$$g \quad \text{is non-decreasing in } x \tag{13}$$

and

$$x \mapsto x - \frac{g(x)}{1+r} \quad \textit{is non-decreasing in } x \tag{14}$$

are satisfied.

Proof. 'If.' Assume that there exists a function $g\colon \mathbb{R}_+ \to \mathbb{R}_+$ such that (2), (3), (9), (13) and (14) are satisfied. By Theorem 1, C satisfies efficiency and time consistency. Let $x > x'$. By (13), we have that

$$g^t(x) \geq g^t(x')$$

for all $t \in \mathbb{Z}_+$. Consequently, since (3) and (14) are satisfied, it follows that

$$\begin{aligned} C_t(x) &= g^t(x) - \frac{g^{t+1}(x)}{1+r} = g^t(x) - \frac{g(g^t(x))}{1+r} \\ &\geq g^t(x') - \frac{g(g^t(x'))}{1+r} = g^t(x') - \frac{g^{t+1}(x')}{1+r} = C_t(x') \end{aligned}$$

for all $t \in \mathbb{Z}_+$. Hence, C satisfies resource monotonicity.

'Only if.' Assume that C satisfies time consistency, efficiency and resource monotonicity. By Theorem 1, there exists a function $g\colon \mathbb{R}_+ \to \mathbb{R}_+$ such that (2), (3) and (9) are satisfied.

To show (13), suppose there exists $x, x' \in \mathbb{R}_+$ such that $x > x'$, but $g(x) < g(x')$. By (3) and (9), it follows that

$$\sum_{t=1}^{\infty} \frac{C_t(x)}{(1+r)^{t-1}} = g(x) < g(x') = \sum_{t=1}^{\infty} \frac{C_t(x')}{(1+r)^{t-1}},$$

contradicting resource monotonicity.

To show (14), suppose there exists $x, x' \in \mathbb{R}_+$ such that $x > x'$, but

$$x - \frac{g(x)}{1+r} < x' - \frac{g(x')}{1+r}.$$

By (3),

$$C_0(x) = x - \frac{g(x)}{1+r} < x' - \frac{g(x')}{1+r} = C_0(x'),$$

again contradicting resource monotonicity. ■

Note that the proof of (13) relies on efficiency, whereas (14) is established without using this axiom.

It follows from Theorems 2 and 3 that the classes of choice functions characterized in Theorem 1 can be narrowed down considerably by adding equity axioms. However, Suppes-Sen or Pigou-Dalton, on the one hand, and resource monotonicity, on the other hand, do so in different ways.

- By Theorem 2, Suppes-Sen or efficiency and Pigou-Dalton in combination with time consistency imply that, for given $x \in \mathbb{R}_+$, $g^t(x)$ and $g^t(x) - (g^{t+1}(x)/(1+r))$ are monotone with respect to t, while
- by Theorem 3, resource monotonicity in combination with efficiency and time consistency imply that $g^t(x)$ and $g^t(x) - (g^{t+1}(x)/(1+r))$ are monotone with respect to x for given $t \in \mathbb{Z}_+$.

5 Examples

To ensure that the choice functions in the examples of this section are well-defined, it is important that the renewal rate r is positive, as we have assumed throughout. Consider first the *steady-state* example, where consumption is equalized across generations.

Example 1. The infinite-horizon choice function C^1 of this example corresponds to the case in which the function g is the identity mapping, defined by $g(x) = x$ for all $x \in \mathbb{R}_+$. This implies $g^t(x) = x$ for all $x \in \mathbb{R}_+$ and for all $t \in \mathbb{Z}_+$. (2) and (9) are satisfied because

$$g(x) = x \leq x(1+r)$$

and

$$\lim_{t\to\infty} \frac{g^t(x)}{(1+r)^t} = \lim_{t\to\infty} \frac{x}{(1+r)^t} = 0$$

for all $x \in \mathbb{R}_+$. According to (3),

$$C_t^1(x) = g^t(x) - \frac{g^{t+1}(x)}{1+r} = x - \frac{x}{1+r} = \frac{xr}{1+r} \tag{15}$$

for all $x \in \mathbb{R}_+$ and for all $t \in \mathbb{Z}_+$, that is, every generation consumes the same amount.

In addition to satisfying time consistency and efficiency, the infinite-horizon choice function C^1 is characterized by a g-function for which the conditions of (11) and (12) hold with equality. By Theorem 2 this entails that C^1 satisfies both Suppes-Sen and Pigou-Dalton. Furthermore, both $g(x)$ and $x - (g(x)/(1+r))$ are non-decreasing in x. Hence, by Theorem 3, the choice function satisfies resource monotonicity, as can easily be verified directly from (15).

A generalization of the choice function C^1 of Example 1 is obtained by letting g be a linear function such that both $g(x)$ and $x - (g(x)/(1+r))$ are non-decreasing in x, so that resource monotonicity is satisfied.

Example 2. The infinite-horizon choice function $C^{2,a}$ of this example is obtained by letting $g(x) = ax$ for all $x \in \mathbb{R}_+$, where $a \in [0, 1+r]$ is a parameter. Obviously, the steady-state case is obtained for $a = 1$. It follows that $g^t(x) = a^t x$ for all $x \in \mathbb{R}_+$ and for all $t \in \mathbb{Z}_+$. Clearly, (2) is satisfied because

$$g(x) = ax \leq x(1+r)$$

for all $x \in \mathbb{R}_+$. (9) is satisfied if and only if $a < 1 + r$ because

$$\lim_{t\to\infty} \frac{g^t(x)}{(1+r)^t} = \lim_{t\to\infty} \frac{a^t x}{(1+r)^t} = \lim_{t\to\infty} \left(\frac{a}{1+r}\right)^t x = 0.$$

Hence, the case where $a = 1 + r$ illustrates how (9) can be violated by excessive accumulation of the resource.

Substituting into (3), it follows that

$$C_t^{2,a}(x) = g^t(x) - \frac{g^{t+1}(x)}{1+r} = a^t x - \frac{a^{t+1}x}{1+r} = \frac{a^t(1+r-a)x}{1+r} \tag{16}$$

for all $x \in \mathbb{R}_+$ and for all $t \in \mathbb{Z}_+$.

In addition to satisfying efficiency and time consistency for $a < 1 + r$, the infinite-horizon choice function $C^{2,a}$ is characterized by a g-function for which the conditions of (11) and (12) hold if and only if $a \geq 1$. By Theorem 2 this entails that $C^{2,a}$ satisfies efficiency, time consistency, Suppes-Sen and Pigou-Dalton if and only if $a \in [1, 1 + r)$. If $a \in (1, 1 + r)$, then consumption is increasing in t, and the consumption of generations t such that

$$t > \frac{\ln(r) - \ln(1+r-a)}{\ln(a)}$$

is higher than that of the steady-state, at the expense of earlier generations. Moreover, the consumption of generation t approaches infinity as t approaches infinity.

Both $g(x)$ and $x - (g(x)/(1+r))$ are non-decreasing in x for any $a \in [0, 1+r]$. Hence, by Theorem 3, the choice function satisfies time consistency, efficiency and resource monotonicity if and only if $a \in [0, 1 + r)$, as can easily be verified directly from (16). Therefore, $C^{2,a}$ satisfies resource monotonicity, but not Suppes-Sen and Pigou-Dalton, if and only if $a \in [0, 1)$. If $a \in (0, 1)$, then consumption is decreasing in t, and the consumption of generations t such that

$$t < \frac{\ln(r) - \ln(1+r-a)}{\ln(a)}$$

is higher than that of the steady-state, at the expense of later generations. Moreover, the consumption of generation t approaches zero as t approaches infinity.

Example 2 shows, in the case where $a < 1$, that $g^t(x)$ and $g^t(x) - (g^{t+1}(x)/(1+r))$ can be non-decreasing with respect to x, without $g^t(x)$ and $g^t(x) - (g^{t+1}(x)/(1 + r))$ being non-decreasing with respect to t. In particular, a choice function can satisfy resource monotonicity without satisfying Suppes-Sen and Pigou-Dalton. In the following pair of examples, we show that a choice function can satisfy Suppes-Sen and Pigou-Dalton without satisfying resource monotonicity.

Example 3. The infinite-horizon choice function C^3 of this example is obtained by setting $r = 1$, so that $1 + r = 2$, and by letting g be given by:

$$g(x) \; = \; \begin{cases} \frac{3}{2}x & \text{if} \quad 0 \leq x \leq 1, \\ \frac{4}{3}x & \text{if} \quad x > 1\,. \end{cases}$$

Clearly, (2) is satisfied. Also, $x \leq g(x)$ for all $x \in \mathbb{R}_+$ so that (11) is satisfied, and $x - g(x)/2$ is an increasing function of x so that (14) is satisfied. By combining these observations we obtain that $x - g(x)/2 \leq g(x) - g^2(x)/2$ for all $x \in \mathbb{R}_+$ so that (12) is satisfied. Furthermore, if $x \in \mathbb{R}_{++}$, then C^3 behaves as $C^{2,a}$ with $a \in (0, 1+r)$ when t goes to infinity, implying that (9) is satisfied. If $x = 0$, then (9) is trivially satisfied. Hence, it follows from Theorem 2 that the infinite-horizon choice function C^3 satisfies efficiency, time consistency, Suppes-Sen and Pigou-Dalton. However,

$$g(1) \;=\; \tfrac{3}{2} \;>\; \tfrac{17}{12} \;=\; g\big(\tfrac{17}{16}\big).$$

Hence, (13) does not hold, and it follows from Theorem 3 that C^3 does not satisfy resource monotonicity.

Example 4. The infinite-horizon choice function C^4 of this example is obtained by setting $r = 1$, so that $1 + r = 2$, and by letting g be given by:

$$g(x) \;=\; \begin{cases} \tfrac{4}{3}x & \text{if} \quad 0 \leq x \leq 1, \\ \tfrac{3}{2}x & \text{if} \quad x > 1. \end{cases}$$

Clearly, (2) is satisfied. Also, $x \leq g(x)$ for all $x \in \mathbb{R}_+$ so that (11) is satisfied, and $g(x)$ is an increasing function of x so that (13) is satisfied. Furthermore, if $x \in \mathbb{R}_{++}$, then C^4 behaves as $C^{2,a}$ with $a \in (0, 1+r)$ when t goes to infinity, implying that (9) is satisfied. If $x = 0$, then (9) is trivially satisfied. To verify that (12) is satisfied, note that

$$\begin{aligned}
x - \tfrac{g(x)}{2} &= \big(1 - \tfrac{2}{3}\big)x = \tfrac{1}{3}x \leq \tfrac{4}{9}x = \big(\tfrac{4}{3} - \tfrac{8}{9}\big)x = g(x) - \tfrac{g^2(x)}{2} && \text{if } 0 \leq x \leq \tfrac{3}{4}, \\
x - \tfrac{g(x)}{2} &= \big(1 - \tfrac{2}{3}\big)x = \tfrac{1}{3}x = \tfrac{1}{3}x = \big(\tfrac{4}{3} - 1\big)x = g(x) - \tfrac{g^2(x)}{2} && \text{if } \tfrac{4}{3} < x \leq 1, \\
x - \tfrac{g(x)}{2} &= \big(1 - \tfrac{3}{4}\big)x = \tfrac{1}{4}x \leq \tfrac{3}{8}x = \big(\tfrac{3}{2} - \tfrac{9}{8}\big)x = g(x) - \tfrac{g^2(x)}{2} && \text{if } x > 1.
\end{aligned}$$

Hence, it follows from Theorem 2 that the infinite-horizon choice function C^4 satisfies efficiency, time consistency, Suppes-Sen and Pigou-Dalton. However,

$$1 - \tfrac{g(1)}{2} \;=\; 1 - \tfrac{2}{3} \;=\; \tfrac{1}{3} \;>\; \tfrac{5}{18} \;=\; \tfrac{10}{9} - \tfrac{5}{6} \;=\; \tfrac{10}{9} - \tfrac{g(10/9)}{2}.$$

Hence, (14) does not hold, and it follows from Theorem 3 that C^4 does not satisfy resource monotonicity.

Examples 2, 3 and 4 show that the conditions characterizing Suppes-Sen and Pigou-Dalton — namely that $g^t(x)$ and $g^t(x) - g^{t+1}(x)/(1+r)$ are monotone with respect to t — are independent of the conditions characterizing resource monotonicity — namely that $g^t(x)$ and $g^t(x) - g^{t+1}(x)/(1+r)$ are monotone with respect to x.

We conclude with an example showing that condition (13) is not necessary for an infinite-horizon choice function to satisfy time consistency and resource monotonicity, as long as efficiency is not imposed.

Example 5. The infinite-horizon choice function C^5 of this example is obtained by setting $r = 1$, so that $1 + r = 2$, and by letting g be given by:

$$g(x) \;=\; \begin{cases} 2x & \text{if} \quad 0 \le x \le 1, \\ 2(x - \frac{1}{2}) & \text{if} \quad x > 1. \end{cases}$$

Clearly (2) is satisfied, while condition (13) is not satisfied, since

$$g(1) \;=\; 2 \;>\; \tfrac{3}{2} \;=\; g\big(\tfrac{5}{4}\big).$$

Resource monotonicity still holds since, by substituting into (3), it follows that

$$C^5(x) \;=\; \begin{cases} (0, 0, \dots) & \text{if} \quad x = 0, \\ \big(\underbrace{0, \dots, 0}_{n+1 \text{ times}}, \frac{1}{2}, \frac{1}{2}, \dots\big) & \text{if} \quad x \in \big((\frac{1}{2})^{n+1}, (\frac{1}{2})^{n}\big] \quad \text{for } n \in \mathbb{Z}_+, \\ \big(\frac{1}{2}, \frac{1}{2}, \dots\big) & \text{if} \quad x > 1. \end{cases}$$

It is straightforward to verify that C^5 does not satisfy efficiency; in particular, increasing the initial resource stock beyond x does not lead to increased consumption for any generation, provided that $x > 1$.

Examples 1 and 2 provide infinite-horizon choice functions that are continuous in the initial endowment, even though there are no continuous orderings satisfying strong Pareto and finite anonymity that rationalize them. This observation serves to further underline the gains that are possible from adopting a choice-theoretic approach.

6 Concluding Remarks

We conclude this chapter with some thoughts on possible directions where the approach of this chapter might be taken in future work. An issue that suggests itself naturally when considering a choice function is its *rationalizability* by a relation defined on the objects of choice — in our case, infinite consumption streams. The rationalizability of choice functions with arbitrary domains has been examined thoroughly in contributions such as Richter [18] and Hansson [13] and, more recently, Bossert, Sprumont and Suzumura [7] and Bossert and Suzumura [9]. While the generality of the results obtained in these papers allow for their application in our intergenerational setting, it might be possible to obtain new observations due to the specific structure of the domain considered here. Note that the existence of a rationalizing ordering does *not* conflict with the impossibility results established for such orderings in the earlier literature: the existence of a rationalization of an infinite-horizon choice function satisfying requirements such as Suppes-Sen does not imply that the choice function is rationalizable by an ordering that possesses properties such as the Suppes-Sen principle formulated for binary relations.

As mentioned earlier, we made the conscious choice to work with a simple model in order to emphasize the novel aspect of the chapter — the choice-theoretic approach in an infinite-horizon setting. It might turn out to be of interest to explore possible generalizations in future work.

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